On the Efficiency and Fairness of Deferred Acceptance With Single Tie-Breaking*

Xiang Han†

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Abstract

As a random allocation rule for indivisible object allocation under weak priorities, deferred acceptance with single tie-breaking (DA-STB) is not ex-post constrained efficient. We first observe that it also fails to satisfy a natural fairness notion, symmetry at the top, which requires that two agents be assigned their common top choice with equal probability if they have equal claim to it. Then, it is shown that DA-STB is ex-post constrained efficient, if and only if it is symmetric at the top, if and only if the priority structure satisfies a certain acyclic condition. We further characterize the priority structures under which DA-STB is ex-post efficient. Based on the characterized priority domains, and using a weak fairness notion called local envy-freeness, new theoretical support is provided for this widely used rule in practice: for any priority structure, among the class of strategy-proof, ex-post stable, symmetric, and locally envy-free rules, each one of the three desiderata—ex-post constrained efficiency, ex-post stability-and-efficiency, and symmetry at the top—can be achieved if and only if it can be achieved by DA-STB.

Keywords: indivisible object; weak priority; random allocation; deferred acceptance with single tie-breaking; ex-post constrained efficiency; school choice

JEL Codes: D47; C78; D82; D78

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†School of Economics, Shanghai University of Finance and Economics, Shanghai 200433, China; Key Laboratory of Mathematical Economics (SUFE), Ministry of Education, Shanghai 200433, China. Email: han.xiang@sufe.edu.cn.

1 Introduction

We consider the priority-augmented indivisible object allocation problem, in which agents have strict preferences over objects, and the objects have priority rankings over the agents. There might be multiple copies of each object, which are to be allocated without monetary transfers. This problem is closely related to the two-sided matching market (Gale and Shapley, 1962), and *school choice* (Abdulkadiroğlu and Sönmez, 2003) is one of its most important applications in practice. The stability concept from the two-sided matching theory is mainly interpreted as a fairness notion in our context: an allocation of the objects is stable if it is individually rational, nonwasteful, and there is no situation in which one agent envies another's assignment for which the first agent has a higher priority.

Preferences of the agents are the only private information, and an allocation rule maps each preference profile of the agents to an allocation, for a given priority structure. In designing a satisfactory rule, fairness, efficiency and incentive compatibility are the main desiderata. When the priorities are strict, the *deferred acceptance algorithm* (DA) from Gale and Shapley (1962) is often considered as the "best rule", which receives some strong theoretical support. First, it is the only stable and strategy-proof rule (Alcalde and Barberà, 1994). Second, it selects the unique stable allocation that Pareto dominates every other stable allocation, i.e., it is stability-constrained efficient. However, it may not be efficient as stability is generally incompatible with efficiency (Roth, 1982, Abdulkadiroğlu and Sönmez, 2003). Full efficiency can be achieved for certain priority structures: Ergin (2002) shows that *acyclicity* on the priority structure is necessary and sufficient for DA to be an efficient rule.[1]

In this paper, we consider the more general setting that allows ties in priorities, which are common in practice.[2] There are two fairness considerations in the context of weak priorities. On one hand, an allocation should respect the differences in priorities in the sense of stability. On the other hand, we want to treat the agents with equal priority for an object in a fair way. Then random allocations are necessary to restore fairness with respect to the ties in priorities. A standard fairness notion in this regard is *equal treatment of equals*, which requires that any two agents with the same preferences and the same priorities for all the objects receive the same lottery. For our purpose,

[1] It follows that acyclicity is also necessary and sufficient for the existence of a stable and efficient rule.

[2] For instance, in Boston school choice programs, students are prioritized based on only two criteria (the sibling and walk zone criteria), and hence large indifference classes exist.

we define a slightly stronger concept: we say a random allocation is *symmetric* if any two agents with the same preferences and the same priorities for their common set of acceptable objects receive the same lottery.

In the presence of ties, there is no longer a "best rule." A natural and common solution is to first break the ties randomly, then apply DA. In fact, DA with *single tie-breaking* (DA-STB) is widely used in practice (Abdulkadiroğlu et al., 2009), in which an ordering of the agents is drawn from the uniform distribution to break the ties at all the objects. While DA-STB is strategy-proof, ex-post stable and symmetric, there is a large class of random allocation rules that satisfy these three basic properties, and it is not clear whether DA-STB has a special role among this class. Moreover, tie-breaking can lead to welfare loss, and DA-STB may not be constrained efficient ex-post (Erdil and Ergin, 2008). In this study we discover a fairness issue that is also related to tie-breaking. One potentially important criterion for welfare evaluation is the agents' probabilities of obtaining their top choices. We define a new concept, *symmetry at the top*, which requires that any two agents with the same top choice and the same priority for this object receive it with equal probability.[3,4] It is then shown that DA-STB is generally not symmetric at the top. In addition, we also introduce *strong symmetry*, which requires that any two agents with the same preferences and priorities for their common top k choices have the same probability of getting each of these k objects, for every $k \geq 1$.

We first want to understand the circumstances in which several desirable efficiency and fairness axioms are satisfied by DA-STB. It is shown that the following four statements are equivalent: (1) DA-STB is ex-post constrained efficient, (2) DA-STB is symmetric at the top, (3) DA-STB is strongly symmetric, and (4) the priority structure is *T-acyclic*. T-acyclicity is defined by ruling out two types of cycles in the priorities that involve ties.[5] When there is such a cycle, for some preferences, applying DA after ties are

[3]The media often focuses on the number of students who receive their top choice when reporting school admissions. For instance, "Higher proportion of pupils fail to get top choice of secondary school" (Weale, 2017), and "According to district officials, only 55% of families to apply for a kindergarten seat during the priority-registration period get a seat in the school they ranked first, while 85% get one of their first three choices" (Larkin and Jung, 2020). Therefore, it is easier to defend a rule if it satisfies symmetry at the top in practice.

[4]In the special case where all the agents have equal claim to each object, symmetry at the top is equivalent to *equal-top fairness* defined by Zhang (2019).

[5]In a cycle of the first type, agent i_1 is ranked higher than two equally ranked agents i_2 and i_3 at object x_1, i_3 is ranked weakly higher than i_4 at $x_2, \dots,$ and i_n is ranked weakly higher than i_1 at x_{n-1}, where $n \geq 3$ and $i_{n+1} = i_1$. In a cycle of the second type, three agents i, j, k are equally ranked at object x, while k is ranked higher than i at another object y. In both cases some scarcity conditions related to object capacities need also be satisfied.

broken by a single ordering will give a constrained inefficient outcome, i.e., breaking ties in the cycle creates artificial stability constraints that lead to welfare loss. Furthermore, Ehlers and Erdil (2010) show that every constrained efficient deterministic allocation for every preference profile is efficient if and only if the priority structure is *strongly acyclic*. We define *strong T-acyclicity* as the combination of T-acyclicity and strong acyclicity. Then it is shown that DA-STB is ex-post efficient if and only if the priority structure is strongly T-acyclic.

Next, we focus on those priority structures that fail to satisfy T-acyclicity or strong T-acyclicity. A natural question to ask is that, given a priority structure under which DA-STB can not deliver a certain desirable efficiency or fairness property (such as ex-post constrained efficiency, ex-post efficiency, symmetry at the top, or strong symmetry), whether some other "reasonable" rule can. We will show that the answer turns out to be no, if a reasonable rule is required to satisfy the three basic properties of strategy-proofness, ex-post stability and symmetry, as well as a new and mild fairness axiom regarding equal priorities.

While symmetry is a common requirement in *house allocation* (Hylland and Zeckhauser, 1979),[6] in our context it is a weak fairness notion regarding equal priorities, since the agents can differ not only in preferences but also in their priority rankings. We introduce *local envy-freeness*, which rules out situations where two agents are ranked equally by some object, the first agent receives this object with probability one, while the second agent, who desires this object, receives it with probability zero. Local envy-freeness can be considered as a very weak version of the original envy-freeness concept in house allocation, tailored to the setting with priorities, and it is satisfied by a large class of rules including DA-STB. Then, as the second set of main results in this paper, we show that if the priority structure is not T-acyclic (resp. strongly T-acyclic), then there does not exist a strategy-proof, ex-post constrained efficient (resp. ex-post stable-and-efficient), symmetric and locally envy-free rule, or a strategy-proof, ex-post stable, symmetric at the top and locally envy-free rule.

These impossibility results highlight the important role of DA-STB: if it fails to satisfy ex-post constrained efficiency, ex-post stability-and-efficiency, symmetry at the top or strong symmetry, then any other "reasonable" rule fails too.[7] Therefore, for exam-

[6]In house allocation, priorities are absent, or equivalently, each object (house) ranks all the agents equally.

[7]It is common to see that impossibility results provide some justification for a rule. As a classical example in the random assignment literature, Bogomolnaia and Moulin (2001) show that, in house allo-

ple, if a market designer values constrained efficiency, and only chooses from strategy-proof, ex-post stable, symmetric and locally envy-free rules, then she would find DA-STB a promising solution without even checking the priority structure, as there is no circumstance where another rule outperforms DA-STB in terms of ex-post constrained efficiency.

As far as we know, despite its popularity in practical applications, there had been almost no theoretical support for DA-STB except the well-known fact that it is strategy-proof, ex-post stable and symmetric. Regarding efficiency, Abdulkadiroğlu et al. (2009) provide some support for DA with a certain fixed single tie-breaking, which is viewed as a deterministic rule. They show that every constrained efficient deterministic allocation can be selected by DA with some fixed single tie-breaking. But this does not imply that DA with some fixed single tie-breaking is a constrained efficient rule. It is also shown that DA with any fixed single tie-breaking cannot be Pareto dominated by a strategy-proof deterministic rule. However, any strategy-proof, individually rational and nonwasteful deterministic rule cannot be Pareto dominated by a strategy-proof deterministic rule (Erdil, 2014, Alva and Manjunath, 2019). On the other hand, regarding fairness, there is in fact a common perception that DA with *multiple tie-breaking* (DA-MTB), in which for each object an ordering of the agents is drawn independently to break the ties, is fairer than DA-STB.[8] However, our results show that DA-STB satisfies symmetry at the top or strong symmetry whenever some other strategy-proof, ex-post stable and locally envy-free rule (such as DA-MTB) does. Moreover, DA-MTB is not symmetric at the top for some T-acyclic priority structures.

1.1 Related Literature

DA-STB was first adopted in school choice programs in New York City and Boston. Abdulkadiroğlu et al. (2005a) and Abdulkadiroğlu et al. (2005b) provide detailed discussions on the practical design considerations in these places. The ex-post constrained inefficiency of DA-STB is not only a theoretical concern: the welfare loss from tie-breaking is significant in realistic scenarios as shown by Erdil and Ergin (2008) using computer

cation, there does not exist a rule that satisfies strategy-proofness, stochastic-dominance-efficiency and equal treatment of equals. Therefore, if we want to achieve the last two properties, then the *probabilistic serial rule* is a good solution, although it is not strategy-proof.

[8]For instance, in the design of NYC high school match DA-MTB was initially preferred to DA-STB by policy makers for fairness reasons (Abdulkadiroğlu et al., 2009).

simulations, and by Abdulkadiroğlu et al. (2009) using field data.[9] Erdil and Ergin (2008) propose an important concept of *stable improvement cycles*, which can be used to Pareto improve the realized outcome of DA-STB, leading to constrained efficiency. However, such procedure is not incentive compatible: they also show that a strategy-proof and constrained efficient deterministic rule generally does not exist. On the other hand, Erdil (2014) considers Pareto improvement over DA-STB from the ex-ante perspective. It is shown that the random allocation selected by DA-STB can be wasteful, and consequently for some priority structures DA-STB admits a strategy-proof improvement in terms of first-order stochastic dominance.[10]

DA-STB may not be the only rule that can be justified using our theoretical results.[11] A general axiomatic characterization of DA-STB is not known yet. As mentioned earlier, in the extreme case of strict priorities DA is characterized by stability and strategy-proofness.[12] This result has been recently strengthened by Alva and Manjunath (2019). They show that DA is the only strategy-proof and *stable-dominating* rule. Ehlers and Klaus (2006) and Kojima and Manea (2010a) also provide characterizations of DA in the context of priority-augmented allocation. In the other extreme case of house allocation, DA-STB is reduced to *random serial dictatorship* (RSD), which is equivalent to core from random endowments (Abdulkadiroğlu and Sönmez, 1998). In addition, Che and Kojima (2010) show that RSD is asymptotically equivalent to the probabilistic serial rule from Bogomolnaia and Moulin (2001). Strategy-proofness, ex-post efficiency and symmetry are well-known properties of RSD. However, it is not clear whether these properties together with some additional ones can pin down RSD. Its axiomatic characterization had been an open question until recently Pycia and Troyan (2022) provide the first characterization using Pareto efficiency, symmetry (as defined in their context) and the key concept of *obvious strategy-proofness*.

There are several studies that investigate the trade-offs between DA-STB and DA-MTB. Empirical evidence by Abdulkadiroğlu et al. (2009) and De Haan et al. (forthcoming) suggests that under DA-STB there are more agents assigned top choices, but also more agents assigned lower ranked objects. Theoretical studies in large markets with

[9] We discuss such evidence in Remark 1 in Section 4.

[10] DA-STB also fails to satisfy several new fairness properties defined from the ex-ante perspective, motivating the design of new random allocation rules: see Kesten and Ünver (2015), Afacan (2018) and Han (2022). However, these properties are generally incompatible with strategy-proofness.

[11] We discuss the uniqueness of DA-STB in Section 5.

[12] Most of the school choice literature studies the case of strict priorities and the rule of DA. See Pathak (2011) for a recent review.

random preferences, such as Ashlagi et al. (2019), Ashlagi and Nikzad (2020), Arnosti (forthcoming) and Allman et al. (forthcoming), compare rank distributions under the two tie-breaking methods and identify circumstances under which STB is superior to MTB.

Finally, following Ergin (2002), there is a growing literature on the characterization of priority structures for some existing rule to satisfy various desirable properties, or for a certain desirable rule to exist. See, for example, Kesten (2006), Ehlers (2007), Haeringer and Klijn (2009), Ehlers and Erdil (2010), Kojima (2011), Kesten (2012), Kojima (2013), Kumano (2013), Chen (2014), Tomoeda (2018), Han (2018), Ehlers and Westkamp (2018) and Ishida (2019). All of these studies focus on deterministic rules. Our results also provide a characterization of the priority structures under which DA with any fixed single tie-breaking is an (constrained) efficient deterministic rule. In the special case of unit capacities, the weak priority structures under which a (group) strategy-proof, stable and efficient deterministic rule exists, and the ones under which a strategy-proof constrained efficient deterministic rule exists, are characterized by Han (2018) and Ehlers and Westkamp (2018) respectively. Generalizations of the results in these two papers to the many-to-one setting remain open questions. Our results provide some partial answers, by restricting attention to random allocation rules that satisfy the fairness properties regarding ties.

2 Preliminaries

Let N be a finite set of **agents** and X a finite set of **objects**. For each object $x \in X$, there are $q_x \geq 1$ copies available, and x has a complete and transitive **priority ordering** $\succeq_x$ on N, with $\succ_x$ and $\sim_x$ denoting its asymmetric and symmetric components, respectively. Given $i \in N$, let $U(\succeq_x, i) = \{j \in N : j \succeq_x i\}$, $SU(\succeq_x, i) = \{j \in N : j \succ_x i\}$, and $I(\succeq_x, i) = \{j \in N : j \sim_x i\}$. A **priority structure** $\succeq = (\succeq_x)_{x \in X}$ is a profile of priority orderings. Let $\emptyset$ denote the null object, or the outside option, with a capacity of $q_\emptyset = +\infty$, and $\bar{X} = X \cup \{\emptyset\}$. Each agent $i \in N$ has a complete, transitive and antisymmetric **preference relation** R_i on $\bar{X}$, with P_i denoting its asymmetric component. An object $x \in X$ is **acceptable** to i if $x R_i \emptyset$. Given $x \in \bar{X}$, let $U(R_i, x) = \left\{y \in \bar{X} : y R_i x\right\}$ and $SU(R_i, x) = \left\{y \in \bar{X} : y P_i x\right\}$. A **preference profile** $R = (R_i)_{i \in N}$ is a list of individual preferences. We fix N, X and $(q_x)_{x \in X}$ in the rest of the paper. Then a **priority-augmented allocation problem**, or simply a **problem**, is represented by a pair $(\succeq, R)$.

A **random allocation**, or simply an **allocation**, is a $|N| \times |\bar{X}|$ matrix M with $M_{ix} \geq 0, \sum_{y \in \bar{X}} M_{iy} = 1$ and $\sum_{j \in N} M_{jx} \leq q_x$ for all $i \in N$ and $x \in \bar{X}$, where M_{ix} represents the probability that i is assigned x. Let $M_i = (M_{ix})_{x \in \bar{X}}$ denote the lottery obtained by i under the allocation M. An allocation M is **deterministic** if $M_{ix} \in \{0, 1\}$ for all $i \in N$ and $x \in \bar{X}$. For ease of exposition, we also use a function $\mu : N \to \bar{X}$, where $|\mu^{-1}(x)| \leq q_x$ for all $x \in \bar{X}$, to represent a deterministic allocation. Let $\mathscr{D}$ be the set of all such deterministic allocations. By a generalized version of the Birkhoff-von Neumann theorem (Birkhoff, 1946, von Neumann, 1953, Kojima and Manea, 2010b), every random allocation can be represented as a lottery over deterministic allocations.[13] That is, for any allocation M, there exists a lottery λ over $\mathscr{D}$ such that $M_{ix} = \sum_{\mu \in \mathscr{D}: \mu(i)=x} \lambda(\mu)$ for all $i \in N$ and $x \in \bar{X}$, where $\lambda(\mu) \geq 0$ denotes the probability of each $\mu \in \mathscr{D}$ and $\sum_{\mu \in \mathscr{D}} \lambda(\mu) = 1$. In this case, we also say that the lottery λ **induces** M. Denote the support of a lottery λ as $\mathscr{S}(\lambda) = \{\mu \in \mathscr{D} : \lambda(\mu) > 0\}$.

Given $(\succeq, R)$, $\mu \in \mathscr{D}$ **Pareto dominates** $\nu \in \mathscr{D}$ if $\mu(i) R_i \nu(i)$ for all $i \in N$ and $\mu(j) P_j \nu(j)$ for some $j \in N$. μ is **efficient** if it cannot be Pareto dominated by any deterministic allocation. μ is **stable** if it satisfies the following conditions: (1) **individual rationality**, $\mu(i) R_i \emptyset$, $\forall i \in N$;[14] (2) **nonwastefulness**, $|\mu^{-1}(x)| < q_x$ implies $\mu(i) R_i x$, $\forall x \in X, i \in N$; (3) **respecting priorities**, $\mu(j) P_i \mu(i)$ implies $j \succeq_{\mu(j)} i$, $\forall i, j \in N$. Then, a random allocation M is **ex-post efficient** (resp., **ex-post stable**) if some λ induces M and each $\mu \in \mathscr{S}(\lambda)$ is efficient (resp., stable). Similarly, M is **ex-post stable-and-efficient** if some λ induces M and each $\mu \in \mathscr{S}(\lambda)$ is both stable and efficient. If M is ex-post stable-and-efficient, then clearly it is both ex-post stable and ex-post efficient. However, the converse might not be true. Han (2015) provides an example in which an ex-post stable and ex-post efficient allocation is not ex-post stable-and-efficient. We include this example in Appendix C.

Given a priority structure $\succeq$, a **rule** is a function that maps each preference profile to a random allocation. A rule f is said to satisfy a certain property defined above if $f(R)$ satisfies this property for all R. f is **strategy-proof** if for each agent truth-telling yields a lottery that first-order stochastically dominates the lottery obtained from reporting any other preferences: for any $i \in N$, $x \in \bar{X}$, R and R'_i, $\sum_{y \in U(R_i, x)} f_{iy}(R) \geq \sum_{y \in U(R_i, x)} f_{iy}(R'_i, R_{-i})$.

[13] There could be multiple lottery representations of a random allocation. Also, see Budish et al. (2013) for a maximal generalization of this theorem to a broader class of allocation problems.

[14] We also say that a random allocation M is individually rational if for all $i \in N$ and $x \in X$, $M_{ix} > 0$ implies $x R_i \emptyset$.

The purpose of using random allocations is to treat agents with equal priorities in a fair way. A standard fairness notion in this regard is *equal treatment of equals*, which requires that any two agents with the same preferences and the same priority at each object should receive the same lottery. For our purpose, we define a slightly stronger version of this notion. Given $(\succeq, R)$, an allocation M is **symmetric** if for any $i, j \in N$ such that $SU(R_i, \emptyset) = SU(R_j, \emptyset)$, $R_i|_{SU(R_i,\emptyset)} = R_j|_{SU(R_i,\emptyset)}$, and $i \sim_x j$ for all $x \in SU(R_i, \emptyset)$, we have $M_i = M_j$.[15] So two agents are considered to be "equals" if they only have different preferences and/or priorities regarding their common set of unacceptable objects. This strengthening of the original equal treatment of equals notion can be justified if we require that the allocation recommended by a rule does not vary with preferences or priorities regarding unacceptable objects.[16]

3 Deferred Acceptance With Single Tie-Breaking

The rule of central interest in this paper is deferred acceptance with single tie-breaking. To introduce this rule, we first consider the special case of strict priorities. That is, consider some $\succeq$ such that $\succeq_x$ is antisymmetric for all $x \in X$. In this case, we can restrict attention to deterministic allocations, since no randomization is needed to deal with the fairness issue regarding ties. Given any R, there exists an *agent-optimal stable* deterministic allocation, i.e., a stable deterministic allocation that Pareto dominates any other stable deterministic allocation. This allocation is given by the following (agent-proposing) **deferred acceptance algorithm** (DA) from Gale and Shapley (1962).

Step 1. Each agent applies to her favorite acceptable object. Each object x places the applicants with the highest priorities up to its quota q_x on its waiting list, and rejects all the other applicants.

Step $k \geq 2$. Each agent who was rejected in Step $k-1$ applies to her next best acceptable object. Each object x chooses among the new applicants and the applicants on its waiting list, places the ones with the highest priorities up to its quota q_x on its waiting list, and rejects all others.

[15] $R_i|_{SU(R_i,\emptyset)}$ is the restriction of R_i to $SU(R_i, \emptyset)$.

[16] In Appendix B, Example B.3 shows that some of our results do not hold if we replace symmetry with equal treatment of equals. However, we can replace it with a slightly weaker requirement: whenever two agents have the same preferences over $\bar{X}$ and the same priorities for their acceptable objects, they receive the same lottery.

The process terminates when there are no more rejections. Then the copies of each object are assigned to the agents on its waiting list.

DA is also strategy-proof (Dubins and Freedman, 1981, Roth, 1982). However, it may not be efficient. Ergin (2002) shows that DA is an efficient rule if and only if $\succeq$ satisfies the *acyclicity* condition, which is defined as the absence of a *cycle* involving three agents and two objects.[17] To facilitate our analysis later, we first present a more general version of this concept. A **generalized cycle** consists of $n \geq 3$ distinct agents $i_1, \dots, i_n \in N$ and $n-1$ distinct objects $x_1, \dots, x_{n-1} \in X$ such that the following two conditions are satisfied:

- (Cycle) $i_1 \succ_{x_1} i_2 \succ_{x_1} i_3$, and $i_{k+1} \succ_{x_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$.
- (Scarcity) There exist $n-1$ (possibly empty) mutually disjoint sets $N_1, \dots, N_{n-1} \subseteq N \setminus \{i_1, \dots, i_n\}$ such that $N_1 \subseteq SU(\succeq_{x_1}, i_2)$, $N_k \subseteq SU(\succeq_{x_k}, i_{k+2})$ for each k with $2 \leq k \leq n-1$, and $|N_k| = q_{x_k} - 1$ for each k with $1 \leq k \leq n-1$.

As shown by Ergin (2002), when there is a generalized cycle, there must also be a generalized cycle that has only three agents, which is simply referred to as a **cycle**. Then the priority structure $\succeq$ is **acyclic** if there does not exist any (generalized) cycle.

Next, consider a weak priority structure $\succeq$. We will also fix this priority structure in the rest of the paper. A natural and common solution in this case is to first break all the ties randomly, then apply DA to the resulting strict priority structure, which gives rise to a random allocation rule. An ordering of the agents can be drawn from the uniform distribution to break the ties at all the objects (*single tie-breaking*). Alternatively, for each object an ordering can be drawn independently to break the ties at this object (*multiple tie-breaking*). Pathak and Sethuraman (2011) introduced a more general method of tie-breaking: we can partition all the objects into a collection of subsets, and conduct a lottery draw independently for each subset.

Formally, an ordering of the agents is a one-to-one function $\sigma : N \to \{1, \dots, |N|\}$. Denote the set of all such orderings as $\mathcal{O}$, and $X = \{x_1, \dots, x_{|X|}\}$. A list of orderings $\varsigma = (\varsigma_{x_1}, \dots, \varsigma_{x_{|X|}}) \in \mathcal{O}^{|X|}$ transforms $\succeq$ into a strict priority structure $\succeq^{\varsigma}$: for all $i, j \in N$ and $x \in X$, $i \succ_x^{\varsigma} j$ if $i \succ_x j$, or, $i \sim_x j$ and $\varsigma_x(i) < \varsigma_x(j)$. Let $f^{DA}(\varsigma, R)$ denote the deterministic allocation obtained from applying DA to $\succeq^{\varsigma}$ and R. Given R, and

[17] He further shows that DA is consistent, or group strategy-proof, if and only if $\succeq$ is acyclic.

a partition $\mathscr{P} = \{X_1, \ldots, X_k\}$ of X, **DA with $\mathscr{P}$-partitioned tie-breaking** selects the random allocation $f^{DA\text{-}\mathscr{P}}(R)$, where for all $i \in N$ and $x \in \bar{X}$,

$$f^{DA\text{-}\mathscr{P}}_{ix}(R) = \frac{1}{(|N|!)^k} \left| \left\{ \varsigma : \varsigma_y = \varsigma_z \ \text{ if } \ y, z \in X_l \ \text{ for some } \ l = 1, \ldots, k, f^{DA}_i(\varsigma, R) = x \right\} \right|.$$

If $|\mathscr{P}| = |X|$, then $f^{DA\text{-}\mathscr{P}} = f^{DA\text{-}MTB}$ is **DA with multiple tie-breaking** (DA-MTB). If $\mathscr{P} = \{X\}$, then $f^{DA\text{-}\mathscr{P}} = f^{DA\text{-}STB}$ is **DA with single tie-breaking** (DA-STB). We will mostly focus on DA-STB. So, for simplicity, denote $\tilde{\mathscr{O}^{|X|}} = \left\{ \varsigma \in \mathscr{O}^{|X|} : \varsigma_x = \varsigma_y \ \text{ for all } \ x, y \in X \right\}$.

In the special case of strict priorities, the use of DA is strongly justified: it is the only stable and strategy-proof rule (Alcalde and Barberà, 1994); it is agent-optimal stable, and hence stability-constrained efficient. In contrast, under weak priorities, there is no such support for DA-STB. First, there is a large class of symmetric, ex-post stable and strategy-proof random allocation rules. For instance, it is easy to see that DA with any partitioned tie-breaking belongs to this class. Second, as shown in Erdil and Ergin (2008), tie-breaking could induce further efficiency loss: for some $\varsigma \in \tilde{\mathscr{O}^{|X|}}$ and R, $f^{DA}(\varsigma, R)$ may be Pareto dominated by another stable deterministic allocation. Therefore, DA-STB may not be even stability-constrained efficient ex-post. Formally, given R, $\mu \in \mathscr{D}$ is **constrained efficient** if it is stable and cannot be Pareto dominated by any stable deterministic allocation. A random allocation M is **ex-post constrained efficient** if some lottery λ induces M and each $\mu \in \mathscr{S}(\lambda)$ is constrained efficient.

Example 1. Suppose that $N = \{1, 2, 3\}$, $X = \{x, y\}$ and $q_x = q_y = 1$. The priority structure $\succeq$ and preference profile R are given as follows.

$$\begin{array}{llcll} x: & 1 \succ_x 2 \sim_x 3 & \qquad & 1: & y, x, \emptyset \\ y: & 1 \sim_y 2 \sim_y 3 & & 2: & x, \emptyset \\ & & & 3: & x, y, \emptyset \end{array}$$

If $(\sigma(1), \sigma(2), \sigma(3)) = (3, 1, 2)$ and $\varsigma_x = \varsigma_y = \sigma$, then $f^{DA}_1(\varsigma, R) = x$. Hence $f^{DA\text{-}STB}_{1x}(R) > 0$. Let λ be any lottery that induces $f^{DA\text{-}STB}(R)$, then there exists $\mu \in \mathscr{S}(\lambda)$ such that $\mu(1) = x$. If μ is nonwasteful and individually rational, then $\mu(2) = \emptyset$ and $\mu(3) = y$. μ is Pareto dominated by the stable deterministic allocation ν with $(\nu(1), \nu(2), \nu(3)) = (y, \emptyset, x)$. Therefore, DA-STB is not ex-post constrained efficient.

In this example, when the ties are broken using ς, a *rejection cycle* emerges in DA: in the first step, agent 2 is tentatively accepted and agent 3 is rejected by x; in the second step agent 3 displaces agent 1 at y; in the third step agent 1 displaces agent 2 at x. This rejection cycle leads to efficiency loss, and a fairness issue as well. Under DA-STB the tie at object x is broken in favor of agent 2 and agent 3 with equal probability. The existence of such a rejection cycle implies that if the tie at object x is broken in favor of agent 2, she may not get x in the end. But note that if the tie is broken in favor of agent 3, then agent 3 always gets x. Therefore, although these two agents have equal claim to their common top choice, they will not be assigned this object with equal probability under DA-STB. In fact, $f_{2x}^{DA\text{-}STB}(R) = \frac{1}{3}$ and $f_{3x}^{DA\text{-}STB}(R) = \frac{1}{2}$.

Definition 1 Given R, an allocation M is **symmetric at the top** if for any $i, j \in N$ and $x \in X$ such that $U(R_i, x) = U(R_j, x) = \{x\}$ and $i \sim_x j$, we have $M_{ix} = M_{jx}$.

An agent's chance of receiving the best outcome is potentially an important measure of her welfare. In the application of school choice, if a group of students consider a certain school as their best choice and they are equally ranked by this school, then it seems unfair to let some of them have better chances of being admitted than the others.

We can also define a stronger fairness notion in this regard: for every $k \geq 1$, if two agents have the same first k choices, and they have the same preferences and priorities for these objects, then they are assigned each of these k objects with equal probability.

Definition 2 Given R, an allocation M is **strongly symmetric** if for any $i, j \in N$ and $x \in X$ such that $\emptyset \notin U(R_i, x) = U(R_j, x)$, $R_i|_{U(R_i,x)} = R_j|_{U(R_i,x)}$ and $i \sim_y j$ for all $y \in U(R_i, x)$, we have $M_{iy} = M_{jy}$ for all $y \in U(R_i, x)$.

Strong symmetry implies symmetry at the top. It also implies symmetry for individually rational allocations. But there is no logical relation between symmetry at the top and symmetry. Regarding the roles of these three notions, it is worth mentioning that symmetry is a minimal requirement that we want a random allocation rule to always satisfy, while the other two are desirable properties that may only be achieved under some circumstances.

4 Results

We are first interested in characterizing the priority structures under which DA-STB satisfies good efficiency or fairness properties. In Example 1, DA-STB is not ex-post constrained efficient or symmetric at the top because of the existence of a certain rejection cycle, which suggests that it fails to satisfy these two properties for similar reasons. As will be formally stated in Theorem 1, it turns out that the priority domain in which DA-STB is ex-post constrained efficient coincides with the one in which it is symmetric at the top or strongly symmetric.[18] This domain is defined by ruling out two types of cycles involving ties.

Definition 3 A **type-I cycle** consists of $n \geq 3$ distinct agents $i_1, \dots, i_n \in N$ and $n-1$ distinct objects $x_1, \dots, x_{n-1} \in X$ such that the following conditions are satisfied:

- (Cycle) $i_1 \succ_{x_1} i_2 \sim_{x_1} i_3$, and $i_{k+1} \succeq_{x_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$.
- (Scarcity) There exist $n-1$ (possibly empty) mutually disjoint sets $N_1, \dots, N_{n-1} \subseteq N \setminus \{i_1, \dots, i_n\}$ such that $N_k \subseteq U(\succeq_{x_k}, i_{k+2})$ and $|N_k| = q_{x_k} - 1$ for each k with $1 \leq k \leq n-1$.[19]

A **type-II cycle** consists of distinct $i, j, k \in N$ and $x, y \in X$ such that the following conditions are satisfied:

- (Cycle) $i \sim_x j \sim_x k \succ_y i$.
- (Scarcity) There exist (possibly empty) disjoint sets $N_x, N_y \subseteq N \setminus \{i, j, k\}$ such that $N_x \subseteq U(\succeq_x, i)$, $N_y \subseteq U(\succeq_y, i)$, $|N_x| = q_x - 1$ and $|N_y| = q_y - 1$.

The priority structure $\succeq$ is **T-acyclic** if there does not exist any type-I or type-II cycle.

In the proof of Theorem 1 below, we will show that a type-I cycle or a type-II cycle will make a general class of rules—which includes DA-STB—fail to satisfy ex-post constrained efficiency or symmetry at the top. To better understand the roles of these two types of cycles for the specific rule, DA-STB, we next provide a characterization of T-acyclicity that relates it to acyclicity from Ergin (2002).

Suppose that for some $\varsigma \in \tilde{\mathcal{O}}^{|X|}$, $\succeq^{\varsigma}$ has a generalized cycle, denoted as c, in which $i_1 \succ^{\varsigma}_{x_1} i_2 \succ^{\varsigma}_{x_1} i_3$, $i_{k+1} \succ^{\varsigma}_{x_k} i_{k+2}$ for $k \in \{2, \dots, n-1\}$, $i_{n+1} = i_1$, and the scarcity condition

[18] Although strong symmetry is stronger than symmetry at the top, the allocation selected by DA-STB is strongly symmetric for every preference profile if the allocation selected by it is symmetric at the top for every preference profile.

[19] Unlike the case of a generalized cycle, when there is a type-I cycle, there may not be a type-I cycle with only three agents.

is satisfied by $N_1, \dots, N_{n-1}$. We will also denote the object x_1 in the generalized cycle c as x_c, and the two agents i_2 and i_3 as i_c and j_c respectively. Define a preference profile R^c as follows.

$$R^c: \quad \begin{array}{ll} i_1: & x_{n-1}, x_1, \emptyset \\ i_2: & x_1, \emptyset \\ i_k: & x_{k-2}, x_{k-1}, \emptyset \quad \text{for } k = 3, \dots, n \\ i \in N_k: & x_k, \emptyset \quad \text{for } k = 1, \dots, n-1 \end{array}$$

In addition, any other agent's first choice is $\emptyset$. Then, the generalized cycle c creates a rejection cycle in DA so that, at $f^{DA}(\varsigma, R^c)$, the agents $i_1, i_3, \dots, i_n$ receive their second choices, and they can perform Pareto-improving exchanges leading to the violation of i_2's priority by i_3 at $\succeq^{\varsigma}_{x_1}$. However, if $i_2 \sim_{x_1} i_3$, such exchanges will not violate any priority under $\succeq$. Therefore, breaking the tie between i_2 and i_3 using ς creates an artificial stability constraint that induces welfare loss, and $f^{DA}(\varsigma, R^c)$ is not constrained efficient for $(\succeq, R^c)$, which suggests that DA-STB is not ex-post constrained efficient. Furthermore, when the tie between i_2 and i_3 is broken in the other way, the previous rejection cycle disappears, which suggests that the two agents do not receive their common top choice x_1 with equal probability under DA-STB.

Therefore, the absence of such generalized cycle in the strict priority structures resulted from single tie-breaking is necessary for DA-STB to be ex-post constrained efficient or symmetric at the top. In fact, it is equivalent to the absence of type-I and type-II cycles:

Proposition 1. *$\succeq$ is T-acyclic if and only if for any $\varsigma \in \tilde{\mathcal{O}}^{|X|}$, $\succeq^{\varsigma}$ does not have a generalized cycle c such that $i_c \sim_{x_c} j_c$.*

Although T-acyclicity is satisfied by any strict priority structure, it imposes strong restrictions on the variations of priority orderings that involve ties, which can be seen from the next proposition. It says that when $\succeq$ is T-acyclic, for an agent i and two objects x and y, if there are at least $q_x + q_y$ other agents who are ranked weakly higher than i by x, and the *priority class* $I(\succeq_x, i)$ has at least three agents, then this priority class is ranked in the same way by both x and y.

Proposition 2. *Suppose that $\succeq$ is T-acyclic. For any $i \in N$ and $x, y \in X$, if $|U(\succeq_x, i)| \geq q_x + q_y + 1$ and $|I(\succeq_x, i)| \geq 3$, then $I(\succeq_x, i) = I(\succeq_y, i)$ and $SU(\succeq_x, i) = SU(\succeq_y, i)$.*

This is similar to Theorem 2 in Ergin (2002), which shows that a strict priority structure is acyclic if and only if the following is true: for any agent i and two objects x and y, if there are at least $q_x + q_y$ agents ranked higher than i by x, then the ranks of i at x and y differ by at most one.

Proposition 2 implies that in practical applications with large priority classes, such as school choice, T-acyclicity is very likely to be violated. In particular, by Proposition 2, T-acyclicity implies that if $|N| > q_x + q_y$ for any two objects x and y, and some object has at least three agents in its bottom priority class, then every object has the same bottom priority class.

Remark 1. *Our theoretical result does not answer the question that whether the violation of T-acyclicity can actually lead to significant welfare loss in practice. When the random allocation selected by DA-STB is not ex-post constrained efficient, it is possible that the DA outcomes are in fact constrained efficient for most orderings used to break ties, i.e., constrained efficient outcomes may realize with sufficiently high probabilities under DA-STB. On the other hand, when T-acyclicity is violated, DA-STB is still ex-post constrained efficient for some preference profiles.*

However, there have been strong evidence for the significant welfare loss under DA-STB in practical school choice problems. Using simulations, Erdil and Ergin (2008) show that DA-STB can be improved significantly through stable improvement cycles. Abdulkadiroğlu et al. (2009) find that, when implementing DA-STB, none of the 250 random draws of the student orderings leads to a constrained efficient outcome using the submitted preference data from the NYC high school admissions in 2006-2007, and only 6% of the random draws leads to constrained efficiency using the data of elementary school applicants in Boston in 2006-2007.

Therefore, given the above discussions, it is also interesting to know that, when T-acyclicity is not satisfied, whether there exists a reasonable rule that outperforms DA-STB. For instance, if the priority structure is not T-acyclic, DA-STB is not ex-post constrained efficient. Then in this case can we find a reasonable rule that is ex-post constrained efficient? We will show that the answer is no, if a reasonable rule must satisfy strategy-proofness, symmetry, as well as an additional and new fairness property that deals with ties.

Recall that symmetry is also a fairness property that deals with ties. However, unlike in house allocation, it is a weak requirement in priority-augmented allocation, since two agents can differ not only in their preferences but also in their priorities. For example, if every two agents have different priority rankings at only one object (that is acceptable to at least one of them), then symmetry does not impose any restriction and every deterministic allocation is symmetric. As a result, we define the following new axiom, which imposes restrictions on the allocation of the probability shares of each object among the agents that are equally ranked by this object.

Definition 4 Given R, an allocation M is **locally envy-free** if there do not exist $i, j \in N$ and $x \in X$ such that $i \sim_x j$, $M_{ix} = 1$, $M_{jx} = 0$ and $\sum_{y \in U(R_j,x)} M_{jy} < 1$.

In the special case of house allocation (i.e., $i \sim_x j$ for all $i, j \in N$ and $x \in X$), an allocation is *envy-free* if for each agent i, her lottery first-order stochastically dominates any other agent's lottery, according to the preferences of i. This is clearly not an appropriate requirement in our model: if M_j does not first-order stochastically dominate M_i according to the preferences of j, then j's (potential) envy may not be "justified" as i may have higher priorities for certain objects. However, if $j \succeq_x i$ for every $x \in X$ with $M_{ix} > 0$, then j's envy is justified and should be eliminated. Local envy-freeness is then defined by ruling out one of the most obvious types of such justified envy. Note that it is hence a weak requirement. As we will use this axiom to prove impossibility results, the weaker it is, the stronger the results are.

Remark 2. *We can also naturally interpret local envy-freeness from the perspective of lottery decompositions. It says that if j can potentially receive y, then it shall not be the case that for any lottery λ that induces M, j weakly envies the object x that i receives (i.e., xP_jy) in all deterministic allocations in the support, $i \sim_x j$, and j never receives x. In addition, this interpretation motivates a stronger concept than local envy-freeness: given R, we say that an allocation M is* ***weakly envy-free*** *if there do not exist $i, j \in N$ and $y \in \bar{X}$ such that for any lottery λ that induces M, $\mu(j) = y$ for some $\mu \in \mathscr{S}(\lambda)$, and for any $\nu \in \mathscr{S}(\lambda)$, we have $\nu(i) \in X$, $\nu(i)P_jy$ and $j \succeq_{\nu(i)} i$. The main results in this paper still hold if local envy-freeness is replaced with weak envy-freeness.*[20]

In many instances local envy-freeness requires a rule to select an allocation that has at least some amount of randomness: if two agents are ranked equally by an object that they both desire, and there is only one copy of this object to be allocated to them,

[20]Notably, Lemma 1 below remains true if we use weak envy-freeness instead of local envy-freeness.

then the rule cannot simply assign this copy to one agent in a deterministic way. Local envy-freeness is satisfied by many common random allocation rules, including DA with any partitioned tie-breaking.

Lemma 1 *$f^{DA\text{-}\mathscr{P}}$ is locally envy-free for any partition $\mathscr{P}$ of X.*

In house allocation, the two most studied random allocation rules, *random serial dictatorship* (RSD) (Abdulkadiroğlu and Sönmez, 1998) and *probabilistic serial rule* (Bogomolnaia and Moulin, 2001), are locally envy-free.[21] Moreover, in allocation problems with weak priorities, several recent studies propose random allocation rules that satisfy new fairness axioms regarding ties in priorities. Local envy-freeness is implied by *no ex-ante discrimination* from Kesten and Ünver (2015) as well as *ordinal fairness* from Han (2022).[22] [23] Although there is no logical relation between local envy-freeness and *claimwise stability* from Afacan (2018), the central claimwise stable rule constructed by him, *constrained probabilistic serial mechanism*, is locally envy-free.[24]

We are ready to present the first main theorem.

Theorem 1 *The following statements are equivalent:*

(i) *DA-STB is ex-post constrained efficient.*

(ii) *DA-STB is strongly symmetric.*

(iii) *There exists a strategy-proof, ex-post constrained efficient, symmetric and locally envy-free rule.*

(iv) *There exists a strategy-proof, ex-post stable, symmetric at the top and locally envy-free rule.*

(v) *$\succeq$ is T-acyclic.*

[21]In fact, the probabilistic serial rule satisfies the stronger notion of envy-freeness, while RSD, which is a special case of DA-STB, is not envy-free.

[22]An allocation M has no ex-ante discrimination if for any $i,j \in N$ and $x \in X$ with $i \sim_x j$, $\sum_{y \in U(R_j,x)} M_{jy} < 1$ implies $M_{jx} \geq M_{ix}$. M is ordinally fair if for any $i,j \in N$ and $x \in X$ with $i \sim_x j$, $M_{ix} > 0$ implies $\sum_{y \in U(R_j,x)} M_{jy} \geq \sum_{y \in U(R_i,x)} M_{iy}$.

[23]Symmetry is also implied by both no ex-ante discrimination and ordinal fairness. In this paper we use two fairness axioms regarding ties in priorities, symmetry and local envy-freeness. This is the weakest set of axioms that we can identify to prove the impossibility results. Moreover, DA-STB does not satisfy no ex-ante discrimination or ordinal fairness, even in the special case of house allocation.

[24]Afacan (2018) studies a more general model in which priorities are random. An allocation M is claimwise stable if for any $i,j \in N$ and $x \in X$, $M_{ix} \leq Pr(i \succ_x j) + \sum_{y \in SU(R_j,x)} M_{jy}$, where $Pr(i \succ_x j)$ is the probability of $i \succ_x j$. Our model can be embedded into his by setting $Pr(i \succ_x j) = \frac{1}{2}$ when $i \sim_x j$.

In the proof of Theorem 1 in Appendix A, we show that for a T-acyclic priority structure, DA-STB is ex-post constrained efficient and strongly symmetric. Then, given that DA-STB is always strategy-proof, ex-post stable, symmetric and locally envy-free, we finish the proof of the theorem by establishing two impossibility results: as long as the priority structure is not T-acyclic, there does not exist a strategy-proof, ex-post constrained efficient, symmetric and locally envy-free rule, or a strategy-proof, ex-post stable, symmetric at the top and locally envy-free rule. Moreover, it can be easily seen that symmetry at the top and strong symmetry are interchangeable in the statements in Theorem 1.

As in almost all the previous studies on characterizations of priority structures, and illustrated by Proposition 2, the condition we found on the priority structure can be quite restrictive (when there are ties). However, the more restrictive this condition is, the stronger our impossibility results are. One may argue that in some practical applications T-acyclicity is most likely not satisfied. When it is not satisfied, DA-STB is not ex-post constrained efficient. Then our impossibility results indicate that this is not only a drawback of DA-STB: any strategy-proof, symmetric and locally envy-free rule is not ex-post constrained efficient. We postpone more detailed and further discussions of the implications of Theorem 1 to Section 5.

Next, to find the priority structures under which DA-STB is ex-post efficient, we only need to complement T-acyclicity with the *strong acyclicity* condition from Ehlers and Erdil (2010). A **weak cycle** consists of distinct $i, j, k \in N$ and $x, y \in X$ such that the following conditions are satisfied: (Cycle) $i \succeq_x j \succ_x k \succeq_y i$; (Scarcity) there exist (possibly empty) disjoint sets $N_x, N_y \subseteq N \setminus \{i, j, k\}$ such that $N_x \subseteq U(\succeq_x, j)$, $N_y \subseteq U(\succeq_y, i)$, $|N_x| = q_x - 1$ and $|N_y| = q_y - 1$. $\succeq$ is **strongly acyclic** if there does not exist any weak cycle.

Definition 5 The priority structure $\succeq$ is **strongly T-acyclic** if it is T-acyclic and strongly acyclic.

Ehlers and Erdil (2010) show that every constrained efficient deterministic allocation under any preference profile is efficient if and only if $\succeq$ is strongly acyclic. Therefore, DA-STB is ex-post stable-and-efficient if $\succeq$ is strongly T-acyclic. The other direction can also be established, through a general (im)possibility result similar to those in Theorem 1.

Theorem 2 *The following statements are equivalent:*

(i) *DA-STB is ex-post stable-and-efficient.*

(ii) *There exists a strategy-proof, ex-post stable-and-efficient, symmetric and locally envy-free rule.*

(iii) $\succeq$ *is strongly T-acyclic.*[25]

Characterization results similar to Proposition 1 can also be obtained for strong acyclicity and strong T-acyclicity. First, there is an interesting connection between strong acyclicity and T-acyclicity: it is straightforward to show that $\succeq$ is strongly acyclic if and only if for any $\varsigma \in \widetilde{\mathcal{O}^{|X|}}$, $\succeq^{\varsigma}$ does not have a cycle c such that $i_c \succ_{x_c} j_c$.[26] Note that if $\succeq^{\varsigma}$ has a cycle c for some $\varsigma \in \widetilde{\mathcal{O}^{|X|}}$, then either $i_c \sim_{x_c} j_c$ or $i_c \succ_{x_c} j_c$. As discussed before, this cycle leads to the constrained inefficiency of $f^{DA}(\varsigma, R^c)$ for $(\succeq, R^c)$ in the former case. In the latter case, $f^{DA}(\varsigma, R^c)$ is constrained efficient for $(\succeq, R^c)$, but it is not efficient. Hence, there is a constrained efficient deterministic allocation that is not efficient. It turns out that ruling out all such cycles in the strict priority structures resulted from single tie-breaking is equivalent to strong acyclicity. Then, combining our characterizations of T-acyclicity and strong acyclicity, it can be easily seen that $\succeq$ is strongly T-acyclic if and only if for any $\varsigma \in \widetilde{\mathcal{O}^{|X|}}$, $\succeq^{\varsigma}$ is acyclic.

Three impossibility results are established in the proofs of Theorem 1 and Theorem 2. In Appendix B, we show that the axioms involved are generally independent, while some stronger impossibility results can be obtained in the special case of one-to-one matching, as explained in the following remark.

Remark 3. *Theorems 1 and 2 include the following two parallel results. First, if $\succeq$ is not T-acyclic, there does not exist a strategy-proof, ex-post constrained efficient, symmetric and locally envy-free rule. Second, if $\succeq$ is not strongly T-acyclic, there does not exist a strategy-proof, ex-post stable-and-efficient, symmetric and locally envy-free rule. In Appendix B, it is shown that they can be strengthened under unit-capacities. Assume $q_x = 1$ for all $x \in X$, then we have:*

- *If $\succeq$ is not T-acyclic, there does not exist a strategy-proof, ex-post constrained efficient and locally envy-free rule.*

[25] Note that, when full efficiency is desired in addition to stability, we want the rule to satisfy ex-post stability-and-efficiency, rather than ex-post stability plus ex-post efficiency. That being said, it can be shown that in the statement (i) of Theorem 2, ex-post stability-and-efficiency can be replaced by ex-post efficiency. Thus, DA-STB is an ex-post stable-and-efficient rule if and only if it is an ex-post efficient (and ex-post stable) rule.

[26] In this statement, "cycle" can also be replaced with "generalized cycle".

- *If $\succeq$ is not strongly T-acyclic, there does not exist an ex-post stable-and-efficient, and locally envy-free rule.*[27]

Therefore, under unit-capacities, symmetry can be dropped from statement (iii) in Theorem 1, and both symmetry and strategy-proofness can be dropped from statement (ii) in Theorem 2.

5 Discussion

We conclude by discussing the main implications of Theorem 1 and Theorem 2.

First, the necessary and sufficient conditions for DA-STB to satisfy several desirable efficiency and fairness axioms are identified. While previous studies on characterizations of priority domains all focus on the deterministic setting, we study one of the most popular random allocation rules. In addition, our results also have implications for deterministic rules. The proof of Theorem 1 indicates that $f^{DA}(\varsigma,\cdot)$ is a constrained efficient deterministic rule for all $\varsigma \in \tilde{\mathscr{O}}^{|X|}$ if and only if $\succeq$ is T-acyclic. Then it follows that $f^{DA}(\varsigma,\cdot)$ is an efficient rule for all $\varsigma \in \tilde{\mathscr{O}}^{|X|}$ if and only if $\succeq$ is strongly T-acyclic.[28] Moreover, Abdulkadiroğlu et al. (2009) show that any constrained efficient allocation can be selected by DA with some fixed single tie-breaking. Therefore, under any preference profile, DA with fixed single tie-breaking characterizes the set of constrained efficient allocations, i.e., for any R the set of constrained efficient allocations is equal to $\left\{f^{DA}(\varsigma,R) : \varsigma \in \tilde{\mathscr{O}}^{|X|}\right\}$, if and only if $\succeq$ is T-acyclic. Similarly, under any preference profile DA with fixed single tie-breaking characterizes the set of stable and efficient allocations if and only if $\succeq$ is strongly T-acyclic.[29] Finally, for the general many-to-one allocation under weak priorities, it is still unknown when a strategy-proof, stable and efficient rule exists, and when a strategy-proof and constrained efficient rule exists.[30] Our results provide some partial answers by restricting attention to random allocation rules that are symmetric and locally envy-free.

Second, despite its popularity in market design applications, a theoretical support

[27]We actually prove these two results under a slightly weaker assumption than unit-capacities, which ensures that every type-I, type-II and weak cycle only involves objects with unit-capacities.

[28]In a related study, Ishida (2019) characterizes the priority structures under which applying DA after the ties in priorities are resolved in any way results in an efficient rule.

[29]This implies a well-known result that in a house allocation problem, serial dictatorships characterize the set of efficient allocations (Svensson, 1994).

[30]Han (2018) and Ehlers and Westkamp (2018) answer these questions in the one-to-one setting, respectively.

for the use of DA-STB, beyond the fact that it is strategy-proof, ex-post stable and symmetric, has been largely elusive, and this paper attempts to fill this gap. If we take strategy-proofness, ex-post stability, symmetry and local envy-freeness as the minimum requirements of a desirable random allocation rule, then ex-post constrained efficiency, ex-post stability-and-efficiency, symmetry at the top or strong symmetry can be achieved if and only if it can be achieved by DA-STB. Therefore, a market designer who values, for instance, constrained efficiency, has a stronger reason to use DA-STB without examining the actual priority structure, since an ex-post constrained efficient rule satisfying those minimum requirements does not exist if DA-STB is not ex-post constrained efficient. To present the theoretical foundations that we provide for DA-STB more clearly, we give additional interpretations in the following remark.

Remark 4. *Recall that a rule was defined for a given priority structure. Now, we abuse the notations slightly and define a rule as a function that selects an allocation for every pair* $(\succeq, R)$*. Then we say that a rule is reasonable if it is strategy-proof, symmetric and locally envy-free (for every priority structure* $\succeq$*).*[31] *Let* $\mathscr{D}$ *denote the collection of all the possible priority structures. Our results imply that, for any reasonable rule* f*, if* $D = \{\succeq \in \mathscr{D} : f^{DA\text{-}STB}(\succeq, R)$ *is ex-post constrained efficient for all* $R\}$ *and* $D' = \{\succeq \in \mathscr{D} : f(\succeq, R)$ *is ex-post constrained efficient for all* $R\}$*, then* $D' \subseteq D$*. That is, DA-STB satisfies ex-post constrained efficiency in a weakly larger priority domain than any other reasonable rule. Ex-post constrained efficiency can be replaced with ex-post stability-and-efficiency in the above statement. It can also be replaced with symmetry at the top or strong symmetry if a reasonable rule is referred to a strategy-proof, ex-post stable and locally envy-free rule.*

For a given priority structure in our characterized priority domains, there can be other rules that perform as well as DA-STB. In particular, when $\succeq$ is T-acyclic, DA-STB may not be the unique rule that satisfies all the axioms in (iii) and (iv) of Theorem 1. We present two representative examples to illustrate that such non-uniqueness is widespread. First, in the special case of house allocation, DA-STB is reduced to RSD. When $|N| \geq 4$, $|X| \geq 3$, and $q_x = 1$ for all $x \in X$, Erdil (2014) constructs a strategy-proof, ex-post efficient and symmetric rule that is different from RSD. In our many-to-one setting, based on his construction, it can be shown that if there are three distinct objects x, y, z such that $|N| > q_x + q_y + q_z$, then there exists a strategy-proof, ex-post

[31] That is, for every $(\succeq, R)$, $f(\succeq, R)$ is symmetric and locally envy-free. Moreover, for every $(\succeq, R)$, $i \in N$ and R'_i, $f_i(\succeq, R)$ first-order stochastically dominates $f_i(\succeq, (R'_i, R_{-i}))$ according to R_i.

efficient, strongly symmetric and locally envy-free rule that is not RSD.[32] Second, the following example shows that even a very small amount of ties in the priority structure can lead to non-uniqueness.

Example 2. Suppose that $|N| \geq 3$, $|X| \geq 2$, and $q_x = 1$ for all $x \in X$. Denote $N = \{1, 2, \ldots, |N|\}$, and let $x, y \in X$ be two distinct objects. The priority structure $\succeq$ is defined such that for all $i, j \in N \setminus \{1, 2, 3\}$ with $i > j$ and $z \in X \setminus \{x, y\}$ we have

$$\begin{aligned} x: &\quad i \succ_x j \succ_x 3 \sim_x 2 \succ_x 1 \\ y: &\quad i \succ_y j \succ_y 3 \sim_y 1 \succ_y 2 \\ z: &\quad i \succ_z j \succ_z 3 \succ_z 2 \succ_z 1 \end{aligned}$$

The priority structure has only two ties and it is T-acyclic. In this case, DA-STB and DA-MTB are not equivalent. To see this, consider any R such that every $i > 3$ ranks $\emptyset$ at the top, and the preferences of 1, 2 and 3 are as follows:

$$\begin{aligned} 1: &\quad y, \emptyset \\ 2: &\quad x, \emptyset \\ 3: &\quad y, x, \emptyset \end{aligned}$$

Then $f^{DA\text{-}STB}_{3x}(R) = \frac{1}{6}$, and $f^{DA\text{-}MTB}_{3x}(R) = \frac{1}{4}$. However, it is easy to check that DA-MTB is also strategy-proof, ex-post constrained efficient, strongly symmetric and locally envy-free under $\succeq$.

In terms of the properties in (iii) and (iv) of Theorem 1, DA-STB is unique in the extreme case of strict priorities, and generally non-unique in the other extreme case of house allocation. Therefore, starting from a strict priority structure, we can add ties into it step by step (in a way that preserves T-acyclicity) to make the priority structure "weaker", until we lose uniqueness. Similarly, starting from the house allocation structure, we can break ties step by step to make it "stricter", and obtain uniqueness at some point. However, it is worth mentioning that generally a "weaker" priority structure does not necessarily indicate that non-uniqueness is more likely: in Example 2, if we add more ties to make $1, 2$ and 3 equally ranked by each object, then DA-STB becomes the unique strategy-proof, ex-post constrained efficient and symmetric rule.[33]

[32] Such a rule is constructed based on RSD, by assigning more probability shares of an object in some particular preference profiles for which RSD is wasteful.

[33] To see that such rule is unique, first note that the agents $N \setminus \{1, 2, 3\}$ always sequentially choose their best available objects in the order of their priorities, due to ex-post stability. Then, the rule allocates the

In the end, although DA-STB is not the only rule that can be justified using our results, a different and systematic rule that can also be supported by the same arguments is not known yet. In other words, other existing rules for priority-augmented allocation cannot be justified in the same way. In particular, DA with any other partitioned tie-breaking does not always satisfy the corresponding desirable efficiency and fairness properties whenever DA-STB does. For instance, we modify the priority structure in Example 1 such that all the three agents are ranked equally by both x and y. If $(\varsigma_x(1), \varsigma_x(2), \varsigma_x(3)) = (1,2,3)$ and $(\varsigma_y(1), \varsigma_y(2), \varsigma_y(3)) = (3,1,2)$, then $f_1^{DA}(\varsigma, R) = x$ and thus $f_{1x}^{DA\text{-}MTB}(R) > 0$. This implies that DA-MTB is not ex-post constrained efficient under this strongly T-acyclic priority structure. Moreover, it is not symmetric at the top: $f_{2x}^{DA\text{-}MTB}(R) = \frac{5}{12}$ and $f_{3x}^{DA\text{-}MTB}(R) = \frac{1}{2}$. Intuitively, compared to single tie-breaking, other tie-breaking methods can lead to more rejection cycles in DA, creating additional issues of efficiency and fairness.

Appendix A: Proofs

Proof of Proposition 1

We start by proving a result regarding longer type-II cycles, which will also be useful in the proofs of other results in the paper. A **generalized type-II cycle** consists of $n \geq 3$ distinct agents $i_1, \ldots, i_n \in N$ and $n-1$ distinct objects $x_1, \ldots, x_{n-1} \in X$ such that the following conditions are satisfied: (Cycle) $i_1 \sim_{x_1} i_2 \sim_{x_1} i_3$, $i_{k+1} \succeq_{x_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, and $i_{k+1} \succ_{x_k} i_{k+2}$ for some k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$; (Scarcity) there exist $n-1$ (possibly empty) mutually disjoint sets $N_1, \ldots, N_{n-1} \subseteq N \setminus \{i_1, \ldots, i_n\}$ such that $N_k \subseteq U(\succeq_{x_k}, i_{k+2})$ and $|N_k| = q_{x_k} - 1$ for each k with $1 \leq k \leq n-1$.

Claim 1. *If there is a generalized type-II cycle, there is a type-I cycle or a type-II cycle.*

Proof. Suppose that there is a generalized type-II cycle. Consider one of the shortest generalized type-II cycles. Assume that it consists of $n \geq 3$ distinct agents $i_1, \ldots, i_n$ and $n-1$ distinct objects $x_1, \ldots, x_{n-1}$, which satisfy the cycle and scarcity conditions in the

remaining objects to $\{1,2,3\}$ in a strategy-proof, ex-post efficient and symmetric way. This implies that the rule allocates these objects in the same way as RSD does, by extending the characterization of RSD in the case of three agents by Bogomolnaia and Moulin (2001).

definition above. If $n=3$, then this is a type-II cycle. Suppose that $n \geq 4$ and consider the following three possible cases.

Case 1. $i_1 \not\succ_{x_{n-1}} i_2$ or $i_1 \not\succ_{x_{n-1}} i_3$. The three agents i_1, i_2, i_3 and the two objects x_1, x_{n-1} constitute a shorter generalized type-II cycle. So this case is not possible.

Case 2. $i_n \succ_{x_{n-1}} i_1 \sim_{x_{n-1}} i_2 \sim_{x_{n-1}} i_3$. Since $i_n \succ_{x_{n-1}} i_1 \sim_{x_{n-1}} i_2$, $i_{k+1} \succeq_{x_k} i_{k+2}$ for $k=1,\ldots,n-2$, and it is easy to check that the scarcity condition is satisfied, the agents $i_n, i_1, \ldots, i_{n-1}$ and the objects $x_{n-1}, x_1, \ldots, x_{n-2}$ constitute a type-I cycle.

Case 3. $i_n \sim_{x_{n-1}} i_1 \sim_{x_{n-1}} i_2 \sim_{x_{n-1}} i_3$. First, $i_n \sim_{x_{n-1}} i_1$ implies that for some k with $2 \leq k \leq n-2$, $i_{k+1} \succ_{x_k} i_{k+2}$. Then, we have $i_n \sim_{x_{n-1}} i_1 \sim_{x_{n-1}} i_3$, and $i_{k+1} \succeq_{x_k} i_{k+2}$ for all k with $2 \leq k \leq n-2$. The scarcity condition can also be easily verified, so a shorter generalized type-II cycle is found, which consists of $n-1$ agents $i_n, i_1, i_3, \ldots, i_{n-1}$ and $n-2$ objects $x_{n-1}, x_2, \ldots, x_{n-2}$. Hence this case is not possible. $\square$

"only if" part. Suppose that for some $\varsigma \in \tilde{\mathcal{O}}^{|X|}$, $\succeq^{\varsigma}$ has a generalized cycle c that consists of distinct $i_1, \ldots, i_n \in N$, $n \geq 3$, and distinct $x_1, \ldots, x_{n-1} \in X$ such that: (1) $i_1 \succ^{\varsigma}_{x_1} i_2 \succ^{\varsigma}_{x_1} i_3$, $i_2 \sim_{x_1} i_3$, and $i_{k+1} \succ^{\varsigma}_{x_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$; (2) there exist $n-1$ mutually disjoint sets $N_1, \ldots, N_{n-1} \subseteq N \setminus \{i_1, \ldots, i_n\}$ such that $N_1 \subseteq SU(\succeq^{\varsigma}_{x_1}, i_2)$, $N_k \subseteq SU(\succeq^{\varsigma}_{x_k}, i_{k+2})$ for each k with $2 \leq k \leq n-1$, and $|N_k| = q_{x_k} - 1$ for each k with $1 \leq k \leq n-1$. Note that $i_1 \succeq_{x_1} i_2$. If $i_1 \succ_{x_1} i_2$, then c becomes a type-I cycle in $\succeq$, and thus $\succeq$ is not T-acyclic. If $i_1 \sim_{x_1} i_2 \sim_{x_1} i_3$, then $\varsigma \in \tilde{\mathcal{O}}^{|X|}$ implies $i_{k+1} \succ_{x_k} i_{k+2}$ for some $k \in \{2, \ldots, n-1\}$. Hence c is a generalized type-II cycle in $\succeq$. By Claim 1, $\succeq$ is not T-acyclic.

"if" part. Suppose that $\succeq$ is not T-acyclic.

Case 1: $\succeq$ has a type-I cycle. Let this type-I cycle consist of distinct $i_1, \ldots, i_n \in N$, $n \geq 3$, and distinct $x_1, \ldots, x_{n-1} \in X$ such that: (1) $i_1 \succ_{x_1} i_2 \sim_{x_1} i_3$, and $i_{k+1} \succeq_{x_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$; (2) there exist $n-1$ mutually disjoint sets $N_1, \ldots, N_{n-1} \subseteq N \setminus \{i_1, \ldots, i_n\}$ such that $N_k \subseteq U(\succeq_{x_k}, i_{k+2})$ and $|N_k| = q_{x_k} - 1$ for each k with $1 \leq k \leq n-1$. Define ς such that $\varsigma_x(j) < \varsigma_x(i_2) < \varsigma_x(i_3) \ldots < \varsigma_x(i_n) < \varsigma_x(i_1)$ for all $x \in X$ and $j \in \cup_{k=1}^{n-1} N_k$. Then, in $\succeq^{\varsigma}$ the above type-I cycle becomes a generalized cycle with $i_2 \sim_{x_1} i_3$.

Case 2: $\succeq$ has a type-II cycle. Let this type-II cycle consist of distinct $i, j, k \in N$ and $x, y \in X$ such that: (1) $i \sim_x j \sim_x k \succ_y i$; (2) there exist disjoint $N_x, N_y \subseteq N \setminus \{i, j, k\}$ such that $N_x \subseteq U(\succeq_x, i)$, $N_y \subseteq U(\succeq_y, i)$, $|N_x| = q_x - 1$ and $|N_y| = q_y - 1$. Define ς such that $\varsigma_z(l) < \varsigma_z(i) < \varsigma_z(j) < \varsigma_z(k)$ for all $z \in X$ and $l \in N_x \cup N_y$. Then the above type-II cycle becomes a cycle in $\succeq^\varsigma$ with $j \sim_x k$. $\square$

Proof of Proposition 2

Suppose that $\succeq$ is T-acyclic. Consider any $i \in N$ and $x, y \in X$ such that $|U(\succeq_x, i)| \geq q_x + q_y + 1$ and $|I(\succeq_x, i)| \geq 3$. We first show that $U(\succeq_x, i) \subseteq U(\succeq_y, i)$. Assume to the contrary, $U(\succeq_x, i) \setminus U(\succeq_y, i) \neq \emptyset$. Pick $j \in U(\succeq_x, i)$ such that $k \succeq_y j$ for all $k \in U(\succeq_x, i)$. Then $i \succ_y j$. As $|I(\succeq_x, i)| \geq 3$, we can find $k \in I(\succeq_x, i)$ such that i, j and k are distinct. Then, we have a type-I cycle with $j \succ_x k \sim_x i \succ_y j$ if $j \succ_x i$, and a type-II cycle with $j \sim_x k \sim_x i \succ_y j$ if $j \sim_x i$. To see that the scarcity condition is satisfied in both cases, note that $|U(\succeq_x, i) \setminus \{i, j, k\}| \geq q_x + q_y - 2$, and $l \succeq_y j$ for all $l \in U(\succeq_x, i) \setminus \{i, j, k\}$ by the initial choice of j. It follows that there exist disjoint $N_x, N_y \subseteq U(\succeq_x, i) \setminus \{i, j, k\}$ such that $N_x \subseteq U(\succeq_x, i)$, $N_y \subseteq U(\succeq_y, j)$, $|N_x| = q_x - 1$ and $|N_y| = q_y - 1$. Therefore, a contradiction is reached, and we have $U(\succeq_x, i) \subseteq U(\succeq_y, i)$.

By the same arguments as before, it can be easily seen that $U(\succeq_x, j) \subseteq U(\succeq_y, j)$ for all $j \in I(\succeq_x, i)$. This indicates that $I(\succeq_x, i) \subseteq I(\succeq_y, i)$.

Given that $|U(\succeq_y, i)| \geq |U(\succeq_x, i)| \geq q_x + q_y + 1$ and $|I(\succeq_y, i)| \geq |I(\succeq_x, i)| \geq 3$, by switching the roles of x and y in all the above arguments, it can be shown that $U(\succeq_y, i) \subseteq U(\succeq_x, i)$ and $I(\succeq_y, i) \subseteq I(\succeq_x, i)$. Therefore, we have $U(\succeq_y, i) = U(\succeq_x, i)$ and $I(\succeq_y, i) = I(\succeq_x, i)$, which imply $SU(\succeq_y, i) = SU(\succeq_x, i)$. $\square$

Proof of Lemma 1

Assume to the contrary, there exist a partition $\mathscr{P}$ of X, R, $i, j \in N$ and $x \in X$ such that $i \sim_x j$, $f^{DA\text{-}\mathscr{P}}_{ix}(R) = 1$, $f^{DA\text{-}\mathscr{P}}_{jx}(R) = 0$ and $\sum_{y \in U(R_j, x)} f^{DA\text{-}\mathscr{P}}_{jy}(R) < 1$. Then for some $\varsigma \in \mathscr{O}^{|X|}$ with $\varsigma_y = \varsigma_z$ for all $X' \in \mathscr{P}$ and $y, z \in X'$, $f^{DA}_i(\varsigma, R) = x$ and $x P_j f^{DA}_j(\varsigma, R)$. By the stability of DA, $\varsigma_x(i) < \varsigma_x(j)$. Next, we adjust the ordering ς_x by moving i down such that i is below j. That is, consider some $\varsigma'_x \in \mathscr{O}$ such that $\varsigma'_x(j) < \varsigma'_x(i)$, and for all $k \in N \setminus \{i\}$ and $l \in N$, $\varsigma'_x(k) < \varsigma'_x(l)$ if $\varsigma_x(k) < \varsigma_x(l)$. Let $\varsigma' = (\varsigma'_x, \varsigma_{-x})$. We want to show that $f^{DA}_i(\varsigma', R) \neq x$. Suppose that this is not true. Then by the stability of DA, $f^{DA}_j(\varsigma', R) R_j x$. Consider any $k, l \in N$ such that $f^{DA}_l(\varsigma', R) P_k f^{DA}_k(\varsigma', R)$. Let $f^{DA}_l(\varsigma', R) = y$. By the stability of DA, $l \succ^{\varsigma'}_y k$. If $y \neq x$, then $l \succ^{\varsigma}_y k$ since $\varsigma_y = \varsigma'_y$. If $y = x$, then $k \neq i$ and the construction of ς'_x implies that we also have $l \succ^{\varsigma}_y k$. So $f^{DA}(\varsigma', R)$ respects the priorities in the problem $(\succeq^\varsigma, R)$. It is also individually rational and nonwasteful for this problem since it is so for $(\succeq^{\varsigma'},R)$. Therefore, $f^{DA}(\varsigma',R)$ is stable for $(\succeq^{\varsigma},R)$. But $f_j^{DA}(\varsigma',R)R_j x P_j f_j^{DA}(\varsigma,R)$, contradicting to the fact that DA is agent-optimal stable. Hence, $f_i^{DA}(\varsigma',R) \neq x$.

Balinski and Sönmez (1999) show that DA *respects improvements*. That is, if an agent's priority ranking at each object weakly increases, she will receive a weakly better object under DA. From $\succeq^{\varsigma}$ to $\succeq^{\varsigma'}$, agent i's ranking at each object weakly decreases, so $f_i^{DA}(\varsigma',R) \neq x$ implies $x P_i f_i^{DA}(\varsigma',R)$. Suppose that $x \in X' \in \mathscr{P}$. Construct ς'' such that for all $z \in X$, $\varsigma''_z = \varsigma'_x$ if $z \in X'$, and $\varsigma''_z = \varsigma_z$ otherwise. From $\succeq^{\varsigma'}$ to $\succeq^{\varsigma''}$, agent i's ranking at each object weakly decreases again, so $x P_i f_i^{DA}(\varsigma'',R)$. This contradicts to the initial assumption of $f_{ix}^{DA\text{-}\mathscr{P}}(R) = 1$. □

Proof of Theorem 1

We first show $(v) \Rightarrow (i)$, then $(v) \Rightarrow (ii)$. It is already known that $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iv)$. So we finish the proof by showing that $(iii) \Rightarrow (v)$ and $(iv) \Rightarrow (v)$.

Proof of *(v)* ⇒ *(i)*. Suppose that there exist R and $\varsigma \in \tilde{\mathscr{O}}^{|X|}$ such that $f^{DA}(\varsigma,R) = \mu$ is not constrained efficient for the problem $(\succeq,R)$. Then, by Erdil and Ergin (2008), there exists a *stable improvement cycle*. That is, there exists a sequence of $n \geq 2$ distinct agents $(i_1,\dots,i_n)$ such that for all $k \in \{1,\dots,n\}$, $\mu(i_k) \in X$, $\mu(i_{k+1})P_{i_k}\mu(i_k)$, and $i_k \succeq_{\mu(i_{k+1})} i$ for all i with $\mu(i_{k+1})P_i\mu(i)$, where $i_{n+1} = i_1$. Without loss of generality, assume that this is one of the shortest stable improvement cycles, and among the shortest ones this cycle minimizes the sum of the involved agents' orderings. That is, in every stable improvement cycle there are at least n agents, and if $(i'_1,\dots,i'_n)$ is a stable improvement cycle, then $\sum_{k=1}^n \sigma(i_k) \leq \sum_{k=1}^n \sigma(i'_k)$, where $\sigma = \varsigma_x$ for all $x \in X$. The objects involved in the cycle $(i_1,\dots,i_n)$ must be distinct. To see this, consider any $k,k' \in \{1,\dots,n\}$ such that $k' > k$. If $k'-k = 1$, then $\mu(i_{k'}) \neq \mu(i_k)$ since $\mu(i_{k'})P_{i_k}\mu(i_k)$. If $k'-k > 1$, then $\mu(i_{k'}) = \mu(i_k)$ implies that $(i_k,\dots,i_{k'-1})$ is a shorter stable improvement cycle.

Define μ' as follows: $\mu'(i) = \mu(i)$ if $i \notin \{i_1,\dots,i_n\}$; $\mu'(i_k) = \mu(i_{k+1})$ for all $k \in \{1,\dots,n\}$. Then μ' is stable for $(\succeq,R)$ and it Pareto dominates $\mu = f^{DA}(\varsigma,R)$. It follows that μ' is not stable for $(\succeq^{\varsigma},R)$. Clearly μ' is individually rational and nonwasteful for $(\succeq^{\varsigma},R)$. Hence, there exist $i,j \in N$ such that $j \succ^{\varsigma}_{\mu'(i)} i$ and $\mu'(i)P_j\mu'(j)$. Then $\mu'(i)P_j\mu(j)$. By the stability of μ for $(\succeq^{\varsigma},R)$, it must be the case that $\mu(i) \neq \mu'(i)$. So $i \in \{i_1,\dots,i_n\}$. Without loss of generality, let $i = i_1$, then $\mu'(i) = \mu(i_2)$. Since $\mu(i_2)P_j\mu(j)$, by the definition of stable improvement cycles, $i_1 \succeq_{\mu(i_2)} j$. Then it follows from $j \succ^{\varsigma}_{\mu(i_2)} i_1$ that $i_1 \sim_{\mu(i_2)} j$ and $\sigma(j) < \sigma(i_1)$.

Next, we show that $\mu(j) \neq \mu(i_k)$ for any $k \in \{1, \dots, n\}$. Given that $\mu(i_2) P_j \mu(j)$, $\mu(j) \neq \mu(i_2)$. Suppose $\mu(j) = \mu(i_1)$. Then $j \notin \{i_2, \dots, i_n\}$ as $\mu(i_1), \dots, \mu(i_n)$ are distinct. Since $i_1 \sim_{\mu(i_2)} j$ and $\mu(i_2) P_j \mu(j)$, we can replace i_1 with j and obtain another stable improvement cycle, $(j, i_2, \dots, i_n)$, with the same length. However, a contradiction is reached since the sum of the involved agents' orderings is smaller in the new stable improvement cycle: $\sum_{k=1}^{n} \sigma(i_k) > \sigma(j) + \sum_{k=2}^{n} \sigma(i_k)$. Finally, suppose that $n \geq 3$ and $\mu(j) = \mu(i_k)$ for some k with $3 \leq k \leq n$. In this case, $j \notin \{i_2, \dots, i_{k-1}\}$, and $(j, i_2, \dots, i_{k-1})$ is a stable improvement cycle with less than n agents, contradiction.

Since μ is stable for $(\succeq, R)$ and $\mu(i_{k+1}) P_{i_k} \mu(i_k)$ for all $k \in \{1, \dots, n\}$, $i_{k+1} \succeq_{\mu(i_{k+1})} i_k$ for all k. For each $k \in \{1, \dots, n\}$, let $N_k = \{l \in N : \mu(l) = \mu(i_k), l \neq i_k\}$. Then by the stability of μ again, $N_{k+1} \subseteq U(\succeq_{\mu(i_{k+1})}, i_k)$ and $|N_{k+1}| = q_{\mu(i_{k+1})} - 1$ for all $k \in \{1, \dots, n\}$, where $N_{n+1} = N_1$. It is obvious that $N_1, \dots, N_n$ are mutually disjoint, and $N_k \subseteq N \setminus \{i_1, \dots, i_n\}$ for each $k \in \{1, \dots, n\}$. Given that $\mu(j) \neq \mu(i_k)$ for any k, $j \notin \{i_1, \dots, i_n\}$ and $j \notin N_k$ for any k.

In sum, we have $n+1$ distinct agents $j, i_1, \dots, i_n$ and n distinct objects $\mu(i_1), \dots, \mu(i_n)$ such that: (1) $i_2 \succeq_{\mu(i_2)} j \sim_{\mu(i_2)} i_1$, $i_{k+1} \succeq_{\mu(i_{k+1})} i_k$ for $k = n, \dots, 2$; (2) there exist n (possibly empty) mutually disjoint sets $N_1, \dots, N_n \subseteq N \setminus \{j, i_1, \dots, i_n\}$ such that $N_{k+1} \subseteq U(\succeq_{\mu(i_{k+1})}, i_k)$ and $|N_{k+1}| = q_{\mu(i_{k+1})} - 1$ for all $k \in \{1, \dots, n\}$. This is a type-I cycle if $i_2 \succ_{\mu(i_2)} i_1$. If $i_2 \sim_{\mu(i_2)} i_1$, then $i_{k+1} \succ_{\mu(i_{k+1})} i_k$ for some $k \in \{2, \dots, n\}$, because otherwise the stability of μ for $(\succeq^{\varsigma}, R)$ implies $\sigma(i_1) > \sigma(i_2) > \dots > \sigma(i_n) > \sigma(i_1)$. Hence, in the case of $i_2 \sim_{\mu(i_2)} i_1$ we obtain a generalized type-II cycle. By Claim 1, there exists a type-I cycle or a type-II cycle.

Therefore, if $\succeq$ is T-acyclic, then for any R' and any $\varsigma' \in \tilde{\mathscr{O}}^{|X|}$, $f^{DA}(\varsigma', R')$ is constrained efficient. It follows that DA-STB is an ex-post constrained efficient rule when $\succeq$ is T-acyclic. $\square$

Proof of *(v)* $\Rightarrow$ *(ii)*. Suppose that $\succeq$ is T-acyclic. Consider any $i, j \in N$, $\bar{x} \in X$ and R such that $\emptyset \notin U(R_i, \bar{x}) = U(R_j, \bar{x}), R_i|_{U(R_i, \bar{x})} = R_j|_{U(R_i, \bar{x})}$ and $i \sim_x j$ for all $x \in U(R_i, \bar{x})$. For each $\varsigma \in \tilde{\mathscr{O}}^{|X|}$, let ς' denote the list of orderings obtained by switching the positions of i and j, i.e., for all $x \in X$, $\varsigma'_x(i) = \varsigma_x(j)$, $\varsigma'_x(j) = \varsigma_x(i)$, and $\varsigma'_x(k) = \varsigma_x(k)$ for all $k \in N \setminus \{i, j\}$. The proof of this direction is mainly built on the following result.

Claim 2. *For any $x \in U(R_i, \bar{x})$ and $\varsigma \in \tilde{\mathscr{O}}^{|X|}$, if $f_i^{DA}(\varsigma, R) = x$ and $x P_j f_j^{DA}(\varsigma, R)$, then $f_j^{DA}(\varsigma', R) R_j x$.*

Proof. Assume to the contrary, there exist $x \in U(R_i, \bar{x})$ and $\varsigma \in \widetilde{\mathcal{O}^{|X|}}$ such that $f_i^{DA}(\varsigma, R) = x$, $xP_j f_j^{DA}(\varsigma, R)$, and $xP_j f_j^{DA}(\varsigma', R)$. Let $f^{DA}(\varsigma, R) = \mu$ and $f^{DA}(\varsigma', R) = \mu'$. Since μ is stable for $(\succeq^{\varsigma}, R)$, $\varsigma_x(i) < \varsigma_x(j)$. Then $\varsigma'_x(j) < \varsigma'_x(i)$. Since μ' is stable for $(\succeq^{\varsigma'}, R)$, $U(R_i, x) = U(R_j, x)$ and $i \sim_y j$ for all $y \in U(R_i, x)$, we have $xP_i\mu'(i)$. Consider two possible cases.

Case 1: $\mu(j)R_j\mu'(j)$.

Define a function $\nu : N \to \bar{X}$ such that for all $k \in N$, $\nu(k) = \mu(k)$ if $\mu(k)R_k\mu'(k)$, and $\nu(k) = \mu'(k)$ if $\mu'(k)P_k\mu(k)$. Then $\nu(i) = x$ and $\nu(j) = \mu(j)$. We first show that ν is a deterministic allocation, i.e., no object is over-assigned. Suppose that there exists $k \in \nu^{-1}(x) \setminus \{i\}$ such that $\mu'(k) = x$. Then $k \succ_x^{\varsigma'} j$, since $xP_j\mu'(j)$ and μ' is stable for $(\succeq^{\varsigma'}, R)$. Because i and j's priority rankings at x are switched as we move from $\succeq^{\varsigma'}$ to $\succeq^{\varsigma}$, it follows that $k \succ_x^{\varsigma} i$. Then $\mu(i) = x$ and the stability of μ for $(\succeq^{\varsigma}, R)$ imply $\mu(k)R_k x$. Then $\mu(k) = x$ since $\nu(k) = x$. Therefore, $\nu^{-1}(x) \subseteq \mu^{-1}(x)$ and $|\nu^{-1}(x)| \leq |\mu^{-1}(x)| \leq q_x$. Next, consider any $y \in X \setminus \{x\}$. y is clearly not over-assigned if $\nu^{-1}(y) \subseteq \mu^{-1}(y)$. Suppose that for some $l \in \nu^{-1}(y)$, $\mu(l) \neq y$. Then $\mu'(l) = y$ and $yP_l\mu(l)$. If there exists $m \in \nu^{-1}(y) \setminus \{l\}$ such that $\mu(m) = y$, then by the stability of μ for $(\succeq^{\varsigma}, R)$, $m \succ_y^{\varsigma} l$. We have $l \notin \{i, j\}$ since $\mu'(l)P_l\mu(l)$, and $m \neq i$ since $\mu(i) = x$. Then $\varsigma'_y(l) = \varsigma_y(l)$, and $\varsigma'_y(m) \leq \varsigma_y(m)$. It follows that $m \succ_y^{\varsigma'} l$. Since μ' is stable for $(\succeq^{\varsigma'}, R)$ and $\mu'(l) = y$, $\mu'(m)R_m y$. Given that $m \in \nu^{-1}(y)$, $\mu'(m) = y$. Therefore, $\nu^{-1}(y) \subseteq (\mu')^{-1}(y)$ and y is not over-assigned. This finishes the proof of the fact that ν is a deterministic allocation.

Consider the problem $(\succeq, R)$. By the construction, ν Pareto dominates the stable deterministic allocation μ'. Then ν is individually rational and nonwasteful.[34] If ν does not respect the priorities, then there exist some $k, l \in N$ such that $k \succ_{\nu(l)} l$ and $\nu(l)P_k\nu(k)$. For some $\mu'' \in \{\mu, \mu'\}$, $\mu''(l) = \nu(l)$. Then $k \succ_{\mu''(l)} l$ and $\mu''(l)P_k\nu(k)R_k\mu''(k)$, contradicting to the stability of μ''. In sum, ν is stable. As ν Pareto dominates μ', μ' is not constrained efficient. However, as shown in the proof of *(v)* $\Rightarrow$ *(i)*, $f^{DA}(\varsigma', \cdot)$ is a constrained efficient rule when $\succeq$ is T-acyclic. So a contradiction is reached.

Case 2: $\mu'(j)P_j\mu(j)$.

[34] Generally, if μ_1 is a stable deterministic allocation and μ_2 Pareto dominates μ_1, then μ_2 is individually rational and nonwasteful. While the individual rationality part is obvious, the nonwastefulness of μ_2 follows from the fact that $|\mu_1^{-1}(x)| = |\mu_2^{-1}(x)|$ for all $x \in X$, which is a part of Lemma 1 in Erdil and Ergin (2008).

It will be shown that there exists a type-I or type-II cycle in this case. First, a finite sequence of $n+1 \geq 2$ distinct agents $(i, i_1, \ldots, i_n)$ is called a *chain* if the following two conditions are satisfied: (1) $\mu'(i_{k+1}) P_{i_k} \mu'(i_k)$ and $\mu'(i_{k+1}) = \mu(i_k)$ for each $k \in \{0, \ldots, n-1\}$, with $i_0 = i$; (2) $\mu'(i) P_{i_n} \mu'(i_n)$ and $\mu'(i) = \mu(i_n)$, or, $\mu'(j) P_{i_n} \mu'(i_n)$ and $\mu'(j) = \mu(i_n)$.

We show that a chain exists. Let $N^W(y) = \left\{l \in \mu^{-1}(y) : y P_l \mu'(l)\right\}$ for each $y \in X$, and $N^W = \{l \in N : \mu(l) P_l \mu'(l)\}$. By the individual rationality of μ', $\mu(l) \in X$ for each $l \in N^W$. Hence, $N^W = \cup_{y \in X} N^W(y)$. For each $y \in X$, the nonwastefulness of μ' implies that $|(\mu')^{-1}(y) \setminus \mu^{-1}(y)| \geq |N^W(y)|$. So there exists a one-to-one function $f : N^W \to N$ such that $\mu'(f(l)) = \mu(l)$ and $\mu(f(l)) \neq \mu(l)$ for all $l \in N^W$. We know that $i \in N^W$ and $\mu'(f(i)) = x$. Since μ' is stable for $(\succeq^{\varsigma'}, R)$ and $x P_j \mu'(j)$, $f(i) \succ_x^{\varsigma'} j$. As i's priority ranking in $\succeq_x^{\varsigma}$ is the same as j's priority ranking in $\succeq_x^{\varsigma'}$, $f(i) \succ_x^{\varsigma} i$. Since $\mu(i) = x$ and μ is stable for $(\succeq^{\varsigma}, R)$, $\mu(f(i)) R_{f(i)} x$. By the construction of f, $\mu(f(i)) \neq x$, so $\mu(f(i)) P_{f(i)} x$. Hence, $f(i) \in N^W$. Next, consider any $l \in N^W \setminus \{i\}$. Since μ' is stable for $(\succeq^{\varsigma'}, R)$, $f(l) \succ_{\mu(l)}^{\varsigma'} l$. We know that $l \neq j$ since $j \notin N^W$. Suppose that $f(l) \neq j$ and $f(l) \neq i$, then $\varsigma_{\mu(l)}(l) = \varsigma'_{\mu(l)}(l)$ and $\varsigma_{\mu(l)}(f(l)) = \varsigma'_{\mu(l)}(f(l))$. Thus $f(l) \succ_{\mu(l)}^{\varsigma} l$. As μ is stable for $(\succeq^{\varsigma}, R)$, $\mu(f(l)) R_{f(l)} \mu(l)$. Then it follows from $\mu(f(l)) \neq \mu(l)$ that $\mu(f(l)) P_{f(l)} \mu(l)$. That is, $f(l) \in N^W$. In sum, it has been shown that f is a one-to-one mapping from N^W to $N^W \cup \{j\}$. This implies that a chain $(i, j_1, \ldots, j_m)$ can be constructed using f, where $j_{k+1} = f(j_k)$ for all $k \in \{0, \ldots, m-1\}$, with $j_0 = i$, and $f(j_m) \in \{i, j\}$.

Let $(i, i_1, \ldots, i_n)$ be the shortest chain. There exists $i^* \in \{i, j\}$ such that $\mu'(i^*) P_{i_n} \mu'(i_n)$ and $\mu'(i^*) = \mu(i_n)$. Let $\{j^*\} = \{i, j\} \setminus \{i^*\}$. We will construct a type-I or type-II cycle using the agents $i = i_0, j, i_1, \ldots, i_n$ and the objects $\mu'(i_1) = x, \ldots, \mu'(i_n), \mu'(i^*)$. We first show that these objects are distinct. Clearly $\mu'(i^*) \neq x$. If $\mu'(i_k) = x$ for some k with $1 < k \leq n$, then $(i, i_k, \ldots, i_n)$ is a shorter chain. Let $i_{n+1} = i^*$. If there exist k and k' such that $1 < k < k' \leq n+1$ and $\mu'(i_k) = \mu'(i_{k'})$, then $(i, i_1, \ldots, i_{k-1}, i_{k'}, \ldots, i_n)$ is a shorter chain when $k' < n+1$, and $(i, i_1, \ldots, i_{k-1})$ is a shorter chain when $k' = n+1$. Hence, $\mu'(i_1), \ldots, \mu'(i_n), \mu'(i^*)$ are distinct. Moreover, $\mu'(j^*) \neq \mu'(i_k)$ for any $k \in \{1, \ldots, n\}$, since $\mu'(j^*) \neq x$ and if $\mu'(j^*) = \mu'(i_k)$ for some k with $2 \leq k \leq n$, then $(i, i_1, \ldots, i_{k-1})$ is a shorter chain. It also follows that $i_1, \ldots, i_n, i^*, j^*$ are distinct. Notice that we cannot rule out the case of $\mu'(i^*) = \mu'(j^*)$.

Case 2.1: $\mu'(i^*) = \mu'(j^*)$, i.e., $\mu'(i) = \mu'(j)$. Since μ' is stable for $(\succeq, R)$ and $\mu'(i) P_{i_n} \mu'(i_n)$, $i \succeq_{\mu'(i)} i_n$ and $j \succeq_{\mu'(i)} i_n$. Given that $\mu'(j) P_j \mu(j)$ and $\mu(i_n) = \mu'(i)$, the stability of μ for $(\succeq, R)$ implies $i_n \succeq_{\mu'(i)} j$. Hence, $i \succeq_{\mu'(i)} i_n \sim_{\mu'(i)} j$. Suppose that $i \sim_{\mu'(i)} i_n \sim_{\mu'(i)} j$. By the stability of μ' for $(\succeq^{\varsigma'}, R)$, $\varsigma'_{\mu'(i)}(i) < \varsigma'_{\mu'(i)}(i_n)$. Then $\varsigma_{\mu'(i)}(j) = \varsigma'_{\mu'(i)}(i) < \varsigma'_{\mu'(i)}(i_n) = \varsigma_{\mu'(i)}(i_n)$. So $j \succ^{\varsigma}_{\mu'(i)} i_n$, contradicting to the stability of μ for $(\succeq^{\varsigma}, R)$. Therefore, we must have $i \succ_{\mu'(i)} i_n \sim_{\mu'(i)} j$. Let $N_{\mu'(i)} = \mu^{-1}(\mu'(i)) \setminus \{i_n\}$ and $N_x = \mu^{-1}(x) \setminus \{i\}$. Obviously, $N_{\mu'(i)}$ and N_x are disjoint. Since $xP_j\mu(j)$ and $\mu'(i)P_j\mu(j)$, we have $N_{\mu'(i)}, N_x \subseteq N \setminus \{i, i_n, j\}$. Then by the stability of μ for $(\succeq, R)$, $N_{\mu'(i)} \subseteq U(\succeq_{\mu'(i)}, j)$, $N_x \subseteq U(\succeq_x, j) = U(\succeq_x, i)$, $|N_{\mu'(i)}| = q_{\mu'(i)} - 1$ and $|N_x| = q_x - 1$. Hence, we have found a type-I cycle, in which $i \succ_{\mu'(i)} i_n \sim_{\mu'(i)} j \succeq_x i$.

Case 2.2: $\mu'(i^*) \neq \mu'(j^*)$. Given that $\mu'(i_{k+1})P_{i_k}\mu'(i_k)$ for $k = 0, \ldots, n$ and μ' is stable for $(\succeq, R)$, we have $i_1 \succeq_x j^* \sim_x i^*$, and $i_{k+1} \succeq_{\mu'(i_{k+1})} i_k$ for $k = n, \ldots, 1$. If $i_1 \sim_x j^* \sim_x i^*$, then $i_{k+1} \succ_{\mu'(i_{k+1})} i_k$ for some $k \in \{1, \ldots, n\}$, since otherwise the stability of μ' for $(\succeq^{\varsigma'}, R)$ implies $\varsigma'_y(i^*) < \varsigma'_y(i_n) < \ldots < \varsigma'_y(i_1) < \varsigma'_y(i^*)$ for any $y \in X$. So the cycle condition of a generalized type-II cycle is satisfied. If $i_1 \succ_x j^* \sim_x i^*$, then the cycle condition of a type-I cycle is satisfied. We finish the proof by establishing the same scarcity condition for either type of cycle. Let $N_k = \{l \in N : \mu'(l) = \mu'(i_k), l \neq i_k\}$ for $k = 1, \ldots, n+1$. Obviously, $N_1, \ldots, N_{n+1}$ are disjoint. Given that $\mu'(j^*) \neq \mu'(i_k)$ for any $k \in \{1, \ldots, n+1\}$, $N_k \subseteq N \setminus \{i^*, j^*, i_1, \ldots, i_n\}$ for all k. Finally, the stability of μ' for $(\succeq, R)$ implies that $N_1 \subseteq U(\succeq_x, i^*)$, $|N_1| = q_x - 1$, $N_{k+1} \subseteq U(\succeq_{\mu'(i_{k+1})}, i_k)$ and $|N_{k+1}| = q_{\mu'(i_{k+1})} - 1$ for $k = n, \ldots, 1$. Therefore, there exists a type-I cycle or a generalized type-II cycle. By Claim 1, there exists a type-I cycle or a type-II cycle.

In sum, it has been shown that in Case 2 there exists a type-I or type-II cycle, contradicting to the initial assumption that $\succeq$ is T-acyclic. $\square$

Consider any $x \in U(R_i, \bar{x})$. To establish the strong symmetry of DA-STB, it is sufficient to show that for any $\varsigma \in \tilde{\mathcal{O}^{|X|}}$, $f_i^{DA}(\varsigma, R) = x$ implies $f_j^{DA}(\varsigma', R) = x$.[35]

Suppose that for some $\varsigma \in \tilde{\mathcal{O}^{|X|}}$, $f_i^{DA}(\varsigma, R) = x$. If $f_j^{DA}(\varsigma, R) = y \in U(R_i, \bar{x})$, then during the process of DA, i and j only apply to some objects in $U(R_i, \bar{x})$, and they only differ in their orderings. Thus, when their orderings are switched, their assignments are also switched. That is, $f_j^{DA}(\varsigma', R) = x$ and $f_i^{DA}(\varsigma', R) = y$.

If $f_j^{DA}(\varsigma, R) = y \notin U(R_i, \bar{x})$, then xP_jy. By Claim 2, $f_j^{DA}(\varsigma', R)R_jx$. Suppose that $f_j^{DA}(\varsigma', R) \neq x$, i.e., $f_j^{DA}(\varsigma', R)P_jx$. Clearly $f_j^{DA}(\varsigma', R) \in U(R_i, \bar{x})$. If $f_i^{DA}(\varsigma', R) \in U(R_i, \bar{x})$, then by a similar argument as before, i and j's assignments are switched when the

[35] If it can be shown that $f_i^{DA}(\varsigma, R) = x$ implies $f_j^{DA}(\varsigma', R) = x$ for any $\varsigma \in \tilde{\mathcal{O}^{|X|}}$, then $\left|\left\{\varsigma \in \tilde{\mathcal{O}^{|X|}} : f_j^{DA}(\varsigma, R) = x\right\}\right| \geq \left|\left\{\varsigma \in \tilde{\mathcal{O}^{|X|}} : f_i^{DA}(\varsigma, R) = x\right\}\right|$ because the mapping $g : \tilde{\mathcal{O}^{|X|}} \to \tilde{\mathcal{O}^{|X|}}$, where $g(\varsigma) = \varsigma'$ for all $\varsigma \in \tilde{\mathcal{O}^{|X|}}$, is one-to-one. By symmetric arguments, $\left|\left\{\varsigma \in \tilde{\mathcal{O}^{|X|}} : f_i^{DA}(\varsigma, R) = x\right\}\right| \geq \left|\left\{\varsigma \in \tilde{\mathcal{O}^{|X|}} : f_j^{DA}(\varsigma, R) = x\right\}\right|$. Hence $f_{ix}^{DA\text{-}STB}(R) = f_{jx}^{DA\text{-}STB}(R)$.

orderings are changed from ς' to ς: $f_i^{DA}(\varsigma,R) = f_j^{DA}(\varsigma',R)$. It follows that $f_i^{DA}(\varsigma,R)P_i x$, contradiction. On the other hand, if $f_i^{DA}(\varsigma',R) \notin U(R_i,\bar{x})$, then $f_j^{DA}(\varsigma',R)P_i f_i^{DA}(\varsigma',R)$. Consider a change of the orderings from ς' to ς. By an argument symmetric to Claim 2, we have $f_i^{DA}(\varsigma,R)R_i f_j^{DA}(\varsigma',R)$. It follows that $f_i^{DA}(\varsigma,R)P_i x$, contradiction. □

Proof of *(iii)* ⇒ *(v)*. Suppose that there exists a strategy-proof, ex-post constrained efficient, symmetric and locally envy-free rule f, but $\succeq$ is not T-acyclic.

Case 1: there exists a type-I cycle.

Let this cycle consist of distinct $i_1,\dots,i_n \in N$, $n \geq 3$, and distinct $x_1,\dots,x_{n-1} \in X$ such that: (1) $i_1 \succ_{x_1} i_2 \sim_{x_1} i_3$, and $i_{k+1} \succeq_{x_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$; (2) there exist $n-1$ mutually disjoint sets $N_1,\dots,N_{n-1} \subseteq N \setminus \{i_1,\dots,i_n\}$ such that $N_k \subseteq U(\succeq_{x_k}, i_{k+2})$ and $|N_k| = q_{x_k} - 1$ for each k with $1 \leq k \leq n-1$. Consider the following two preference profiles.[36]

$$R: \quad \begin{array}{ll} i_1: & x_{n-1}, x_1, \emptyset \\ i_2: & x_1, \emptyset \\ i_k: & x_{k-2}, x_{k-1}, \emptyset \quad \text{for } k = 3,\dots,n \\ i \in N_k: & x_k, \emptyset \quad \text{for } k = 1,\dots,n-1 \end{array}$$

$$R': \quad \begin{array}{ll} i_1: & x_{n-1}, \emptyset \\ i_2: & x_1, \emptyset \\ i_k: & x_{k-2}, \emptyset \quad \text{for } k = 3,\dots,n \\ i \in N_k: & x_k, \emptyset \quad \text{for } k = 1,\dots,n-1 \end{array}$$

Since f is ex-post constrained efficient, there exists a lottery λ that induces $f(R)$ and each $\mu \in \mathscr{S}(\lambda)$ is constrained efficient.

First, we show that $f_{i_1x_1}(R) = 0$. Assume to the contrary, $f_{i_1x_1}(R) > 0$. Then for some $\mu \in \mathscr{S}(\lambda)$, $\mu(i_1) = x_1$. If $\mu(i_2) = x_1$, then there exists $i \in N_1$ such that $\mu(i) = \emptyset$. By stability, $i \sim_{x_1} i_2 \sim_{x_1} i_3$. The other possible case is that $\mu(i_2) \neq x_1$. Therefore, in sum, there must exist some $j \in \{i_2\} \cup N_1$ such that $\mu(j) = \emptyset$ and $j \sim_{x_1} i_3$. Consider the deterministic allocation ν in which $\nu(j) = \emptyset$ and every agent except j is assigned her top choice. ν is well defined given the construction of R. It follows from $j \sim_{x_1} i_3$ that

[36] Any unlisted agent is assumed to rank her outside option at the top. Then such an agent will be assigned her outside option with probability one under an individually rational rule.

$j' \succeq_{x_1} j$ for all $j' \in \nu^{-1}(x_1)$. So ν is stable. A contradiction is reached since ν Pareto dominates the constrained efficient deterministic allocation μ.

Second, we show that $f_{i_1 x_{n-1}}(R) < 1$. Assume to the contrary, $f_{i_1 x_{n-1}}(R) = 1$. Then by strategy-proofness, $f_{i_1 x_{n-1}}(R'_{i_1}, R_{-i_1}) = 1$. Consider any $i \in N_{n-1}$. If $i \succ_{x_{n-1}} i_1$, then by ex-post stability, $f_{i x_{n-1}}(R'_{i_1}, R_{-i_1}) = 1$. If $i \sim_{x_{n-1}} i_1$, then by symmetry, $f_{i x_{n-1}}(R'_{i_1}, R_{-i_1}) = 1$. It follows that $f_{i_n x_{n-1}}(R'_{i_1}, R_{-i_1}) = 0$. Since $i_n \succeq_{x_{n-1}} i_1$, $i_n \succ_{x_{n-1}} i_1$ or $i_n \sim_{x_{n-1}} i_1$. In the former case, by ex-post stability, $f_{i_n x_{n-2}}(R'_{i_1}, R_{-i_1}) = 1$. In the latter case, by local envy-freeness, we also have $f_{i_n x_{n-2}}(R'_{i_1}, R_{-i_1}) = 1$. By similar arguments, it can be shown that $f_{i_n x_{n-2}}(R'_{i_1}, R_{-i_1}) = 1$ implies $f_{i_n x_{n-2}}(R'_{i_1}, R'_{i_n}, R_{-\{i_1, i_n\}}) = 1$, which further implies $f_{i_{n-1} x_{n-3}}(R'_{i_1}, R'_{i_n}, R_{-\{i_1, i_n\}}) = 1$ (when $n > 3$). Continuing in this fashion, eventually we have $f_{i_3 x_1}(R') = 1$. Then by symmetry and ex-post stability, $f_{i_2 x_1}(R') = 1$ and $f_{i x_1}(R') = 1$ for all $i \in N_1$. A contradiction is reached since at least $q_{x_1} + 1$ agents are assigned x_1 with probability one.

Finally, given that $f_{i_1 x_1}(R) = 0$ and $f_{i_1 x_{n-1}}(R) < 1$, $f_{i_1 \emptyset}(R) > 0$. So for some $\mu' \in \mathscr{S}(\lambda)$, $\mu'(i_1) = \emptyset$. Since μ' is stable and $i_1 \succ_{x_1} i_2 \sim_{x_1} i_3$, $\mu'(i_2) \neq x_1$ and $\mu'(i_3) \neq x_1$. It follows that $(\mu')^{-1}(x_1) \subseteq N_1$, contradicting to the nonwastefulness of μ'.

Case 2: there exists a type-II cycle.

We first refine the notion of a type-II cycle. A **type-II* cycle** consists of distinct $i, j, k \in N$ and $x, y \in X$ such that the following conditions are satisfied: (Cycle) $i \sim_x j \sim_x k \succ_y i$ and $j \succ_y i$; (Scarcity) there exist (possibly empty) disjoint sets $N_x, N_y \subseteq N \setminus \{i, j, k\}$ such that $N_x \subseteq U(\succeq_x, i)$, $N_x \subseteq SU(\succeq_y, i)$, $N_y \subseteq SU(\succeq_y, i)$, $|N_x| = q_x - 1$ and $|N_y| = q_y - 1$.

Claim 3. *If there is a type-II cycle but not any type-I cycle, then there is a type-II* cycle.*

Proof. Suppose that there does not exist any type-I cycle, and there exists a type-II cycle that consists of distinct $i, j, k \in N$ and $x, y \in X$ such that: (1) $i \sim_x j \sim_x k \succ_y i$; (2) there exist disjoint sets $N_x, N_y \subseteq N \setminus \{i, j, k\}$ such that $N_x \subseteq U(\succeq_x, i)$, $N_y \subseteq U(\succeq_y, i)$, $|N_x| = q_x - 1$ and $|N_y| = q_y - 1$. If $j \sim_y i$, then there is a type-I cycle since $k \succ_y i \sim_y j \succeq_x k$, $N_y \subseteq U(\succeq_y, i)$ and $N_x \subseteq U(\succeq_x, k)$. So we have either $j \succ_y i$ or $i \succ_y j$. Suppose $j \succ_y i$. We will show that a type-II* cycle exists in this case. The other case can be shown similarly by switching the roles of i and j.

First, $N_y \subseteq SU(\succeq_y, i)$. If this is not true, there exists $l \in N_y$ such that $l \sim_y i$. A type-I cycle is found since $k \succ_y l \sim_y i \succeq_x k$ and the scarcity condition is satisfied by $(N_y \setminus \{l\}) \cup \{j\}$ and N_x.

Second, if $l \in N_x$ and $l \succ_x i$, $l \succ_y i$. To see this, suppose $l \in N_x$, $l \succ_x i$ and $i \succeq_y l$. Then there is a type-I cycle: $l \succ_x j \sim_x i \succeq_y l$ and the scarcity condition is satisfied by $(N_x \setminus \{l\}) \cup \{k\}$ and N_y.

Third, if $l \in N_x$ and $l \sim_x i$, $l \not\sim_y i$. If this is not true, there is a type-I cycle: $k \succ_y l \sim_y i \succeq_x k$ and the scarcity condition is satisfied by N_y and $(N_x \setminus \{l\}) \cup \{j\}$.

Fourth, if $l, l' \in N_x$, $l \neq l'$, $l \sim_x l' \sim_x i$, $i \succ_y l$ and $i \succ_y l'$, then $l \not\sim_y l'$. If this is not true, there is a type-I cycle: $i \succ_y l \sim_y l' \succeq_x i$ and the scarcity condition is satisfied by N_y and $(N_x \setminus \{l, l'\}) \cup \{j, k\}$.

Therefore, if $N^* = \left\{l \in N_x : l \sim_x i, i \succ_y l\right\}$ is empty, then by the first three results the type-II cycle that consists of $i, j, k \in N$ and $x, y \in X$ is also a type-II* cycle. If N^* is nonempty, by the fourth result, we can find $l \in N^*$ such that $l' \succ_y l$ for any $l' \in N^* \setminus \{l\}$. Then i, j, l and x, y form a type-II* cycle: $l \sim_x i \sim_x j \succ_y l$, $i \succ_y l$, and the scarcity condition is satisfied by $(N_x \setminus \{l\}) \cup \{k\}$ and N_y. $\square$

In light of Claim 3, it is sufficient to consider a type-II* cycle. Let this cycle consist of $i, j, k \in N$ and $x, y \in X$ such that: (1) $i \sim_x j \sim_x k \succ_y i$ and $j \succ_y i$; (2) there exist disjoint $N_x, N_y \subseteq N \setminus \{i, j, k\}$ such that $N_x \subseteq U(\succeq_x, i)$, $N_x \subseteq SU(\succeq_y, i)$, $N_y \subseteq SU(\succeq_y, i)$, $|N_x| = q_x - 1$ and $|N_y| = q_y - 1$. Let $N_x^* = \{l \in N_x : l \sim_x i\}$ and $q = q_x - |N_x \setminus N_x^*|$. Then $|N_x^*| = q - 1$. In the following analysis, we will restrict attention to preference profiles in which x is only acceptable to some agents in $\{i, j, k\} \cup N_x$, and the agents in $N_x \setminus N_x^*$ rank x at the top. By the ex-post stability of f, each agent in $N_x \setminus N_x^*$ is always assigned x with probability one. Hence, $q \geq 1$ is essentially the number of copies of x available for $\{i, j, k\} \cup N_x^*$. Consider the following preference profiles.

$R^1:$	$i:$	$x, \emptyset$	$R^2:$	$i:$	$x, y, \emptyset$	$R^3:$	$i:$	$y, x, \emptyset$
	$j:$	$x, \emptyset$		$j:$	$x, y, \emptyset$		$j:$	$x, \emptyset$
	$k:$	$x, \emptyset$		$k:$	$x, y, \emptyset$		$k:$	$x, y, \emptyset$
	$N_x^*:$	$x, \emptyset$		$N_x^*:$	$x, y, \emptyset$		$N_x^*:$	$x, y, \emptyset$
	$N_y:$	$y, \emptyset$		$N_y:$	$y, \emptyset$		$N_y:$	$y, \emptyset$
	$N_x \setminus N_x^*:$	$x, \emptyset$		$N_x \setminus N_x^*:$	$x, \emptyset$		$N_x \setminus N_x^*:$	$x, \emptyset$

At R^1, by individual rationality and ex-post nonwastefulness, $\sum_{l \in \{i,j,k\} \cup N_x^*} f_{lx}(R^1) = q$.[37] Then by symmetry, $f_{lx}(R^1) = \frac{q}{q+2}$ for all $l \in \{i, j, k\} \cup N_x^*$. We use induction to show that the allocation of x is the same at R^2. Consider the following statement: for any $N' \subseteq$

[37]Formally, an allocation M is ex-post nonwasteful if some lottery λ induces M and each $\mu \in \mathscr{S}(\lambda)$ is nonwasteful. It is worth mentioning that an ex-post nonwasteful allocation M may involve "waste"

$\{i,j,k\} \cup N_x^*$ with $|N'| = n$, $f_{lx}(R^2_{N'}, R^1_{-N'}) = \frac{q}{q+2}$ for all $l \in \{i,j,k\} \cup N_x^*$. It is obviously true if $n = 0$. Suppose that the statement is true for n with $0 \le n \le q+1$. Consider any $N' \subseteq \{i,j,k\} \cup N_x^*$ with $|N'| = n+1$ and any $l \in N'$. Then $f_{lx}(R^2_{N'\setminus\{l\}}, R^1_{-N'\setminus\{l\}}) = \frac{q}{q+2}$. By strategy-proofness, $f_{lx}(R^2_{N'}, R^1_{-N'}) = \frac{q}{q+2}$. Notice that the choice of l is arbitrary within the set N'. So $f_{l'x}(R^2_{N'}, R^1_{-N'}) = \frac{q}{q+2}$ for all $l' \in N'$. By individual rationality and ex-post nonwastefulness, $\sum_{l' \in (\{i,j,k\} \cup N_x^*)\setminus N'} f_{l'x}(R^2_{N'}, R^1_{-N'}) = q - \frac{q}{q+2}|N'|$. Then the symmetry of f with respect to $(\{i,j,k\} \cup N_x^*) \setminus N'$ implies $f_{l'x}(R^2_{N'}, R^1_{-N'}) = \frac{q}{q+2}$ for all $l' \in \{i,j,k\} \cup N_x^*$. That is, the statement is true for $n+1$. Therefore, $f_{l'x}(R^2) = \frac{q}{q+2}$ for all $l' \in \{i,j,k\} \cup N_x^*$.

Let μ be any stable deterministic allocation for $(\succeq, R^2)$. Since $l \succ_y i$ for all $l \in \{j,k\} \cup N_x^* \cup N_y$, if $\mu(i) = y$, then $\mu(l) = y$ for all $l \in N_y$, and $\mu(l) = x$ for all $l \in \{j,k\} \cup N_x^*$, which is not possible. So by ex-post stability, $f_{iy}(R^2) = 0$. By the same reasoning, $f_{iy}(R^3_i, R^2_{-i}) = 0$. Then the strategy-proofness of f implies $f_{ix}(R^3_i, R^2_{-i}) = f_{ix}(R^2) = \frac{q}{q+2}$. It follows that for some $l \in \{j,k\} \cup N_x^*$, $f_{lx}(R^3_i, R^2_{-i}) \le \frac{q}{q+2}$. Without loss of generality, suppose that $f_{jx}(R^3_i, R^2_{-i}) \le \frac{q}{q+2}$. Then by strategy-proofness,

$$f_{jx}(R^3) = f_{jx}(R^3_i, R^2_{-i}) \le \frac{q}{q+2}. \tag{1}$$

Consider R_3. We argue that $f_{iy}(R^3) + f_{jx}(R^3) \ge 1$. Assume to the contrary, $f_{iy}(R^3) + f_{jx}(R^3) < 1$. Let λ be a lottery that induces $f(R^3)$ such that each $\mu \in \mathscr{S}(\lambda)$ is constrained efficient. Then there exists $\mu \in \mathscr{S}(\lambda)$ such that $\mu(i) \neq y$ and $\mu(j) \neq x$. By individual rationality, $\mu(j) = \emptyset$. Consider the deterministic allocation ν in which $\nu(j) = \emptyset$ and every other agent is assigned her top choice. It is easy to see that ν is stable and it Pareto dominates μ, contradicting to the constrained efficiency of μ. Therefore, $f_{iy}(R^3) + f_{jx}(R^3) \ge 1$. Then it follows from (1) that

$$f_{iy}(R^3) \ge \frac{2}{q+2}. \tag{2}$$

Finally, consider the following preference profiles.

ex-ante, i.e., there may exist $i \in N$ and $x \in X$ such that $\sum_{y \in U(R_i,x)} M_{iy} < 1$ and $\sum_{j \in N} M_{jx} < q_x$. However, it can be easily shown that generally if an allocation M is ex-post nonwasteful and there are at least q_x agents rank some object $x \in X$ at the top, then $\sum_{i \in N} M_{ix} = q_x$.

$$R^4: \begin{array}{ll} i: & y, \emptyset \\ j: & x, \emptyset \\ k: & x, \emptyset \\ N_x^*: & x, \emptyset \\ N_y: & y, \emptyset \\ N_x \setminus N_x^*: & x, \emptyset \end{array} \qquad R^5: \begin{array}{ll} i: & y, \emptyset \\ j: & x, \emptyset \\ k: & x, y, \emptyset \\ N_x^*: & x, y, \emptyset \\ N_y: & y, \emptyset \\ N_x \setminus N_x^*: & x, \emptyset \end{array}$$

At R^4, by individual rationality and ex-post nonwastefulness, $\sum_{l \in \{j,k\} \cup N_x^*} f_{lx}(R^4) = q$. Then by symmetry, $f_{lx}(R^4) = \frac{q}{q+1}$ for all $l \in \{j,k\} \cup N_x^*$. Similar to the previous discussion about moving from R^1 to R^2, we can use induction to show that $f_{lx}(R^5) = \frac{q}{q+1}$ for all $l \in \{j,k\} \cup N_x^*$. Next, consider any stable deterministic allocation μ for $(\succeq, R^5)$ and any $l \in \{k\} \cup N_x^* \cup N_y$. Suppose that $\mu(l) = \emptyset$. Then by stability $\mu(i) = \emptyset$. Since the q copies of x and q_y copies of y can only be allocated to $\{i,j,k\} \cup N_x^* \cup N_y$ at μ and $|\{i,j,k\} \cup N_x^* \cup N_y| = q + q_y + 1$, there is at least one copy of x or one copy of y that is unassigned. This implies that μ is wasteful because $q+1$ agents in $\{i,j,k\} \cup N_x^* \cup N_y$ rank x at the top and q_y agents in this set rank y at the top. Therefore, $\mu(l') \neq \emptyset$ for any $l' \in \{k\} \cup N_x^* \cup N_y$. Then by ex-post stability, $f_{l'y}(R^5) = 1$ for all $l' \in N_y$, and $f_{l'y}(R^5) = 1 - \frac{q}{q+1} = \frac{1}{q+1}$ for all $l' \in \{k\} \cup N_x^*$. Moreover, $\sum_{l' \in \{i,k\} \cup N_x^* \cup N_y} f_{l'y}(R^5) = q_y$. Hence,

$$\begin{aligned} f_{iy}(R^5) &= q_y - \sum_{l' \in \{k\} \cup N_x^*} f_{l'y}(R^5) - \sum_{l' \in N_y} f_{l'y}(R^5) \\ &= q_y - \frac{q}{q+1} - (q_y - 1) \\ &= \frac{1}{q+1} \\ &< \frac{2}{q+2} \end{aligned} \tag{3}$$

Comparing (2) and (3), f is not strategy-proof, contradiction. □

Proof of *(iv)* ⇒ *(v)*. Suppose that there exists a strategy-proof, ex-post stable, symmetric at the top and locally envy-free rule f, but $\succeq$ is not T-acyclic. In light of Claim 3, we consider the following two cases.

Case 1: There is a type-I cycle.

Let this cycle consist of distinct $i_1, \ldots, i_n \in N$, $n \geq 3$, and distinct $x_1, \ldots, x_{n-1} \in X$

such that: (1) $i_1 \succ_{x_1} i_2 \sim_{x_1} i_3$, and $i_{k+1} \succeq_{x_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$; (2) there exist $n-1$ mutually disjoint sets $N_1, \ldots, N_{n-1} \subseteq N \setminus \{i_1, \ldots, i_n\}$ such that $N_k \subseteq U(\succeq_{x_k}, i_{k+2})$ and $|N_k| = q_{x_k} - 1$ for each k with $1 \leq k \leq n-1$. Consider the following preference profile.

$$R: \quad \begin{array}{ll} i_1: & x_{n-1}, x_1, \emptyset \\ i_2: & x_1, \emptyset \\ i_3: & x_1, \emptyset \\ i_k: & x_{k-2}, x_{k-1}, \emptyset \quad \text{for } k = 4, \ldots, n \ \text{(if } n \geq 4) \\ N_k: & x_k, \emptyset \quad \text{for } k = 1, \ldots, n-1 \end{array}$$

Denote $N_1^* = \{i \in N_1 : i \sim_{x_1} i_2\}$. Let μ be any stable deterministic allocation for $(\succeq, R)$. Then $\mu(i) = x_1$ for all $i \in N_1 \setminus N_1^*$. If $n = 3$, clearly $\mu(i_1) = x_2$. Suppose $n \geq 4$. By nonwastefulness and individual rationality, $\mu(i_4) = x_2$, then $\mu(i_5) = x_3, \ldots, \mu(i_{n+1}) = x_{n-1}$. Therefore, by ex-post stability, $f_{i_1 x_{n-1}}(R) = 1$ and $f_{i_1 x_1}(R) = 0$. Moreover, $f_{i x_1}(R) = 1$ for all $i \in N_1 \setminus N_1^*$, and $\sum_{i \in \{i_2, i_3\} \cup N_1^*} f_{i x_1}(R) = q_{x_1} - |N_1 \setminus N_1^*| \geq 1$. It follows from symmetry at the top that $f_{i x_1}(R) = f_{i_3 x_1}(R)$ for all $i \in \{i_2, i_3\} \cup N_1^*$.

Next, define $R'_{i_3} : x_1, x_2, \emptyset$. Let $R' = (R'_{i_3}, R_{-i_3})$. By strategy-proofness, $f_{i_3 x_1}(R') = f_{i_3 x_1}(R)$. By symmetry at the top, $f_{i x_1}(R') = f_{i_3 x_1}(R)$ for all $i \in \{i_2, i_3\} \cup N_1^*$. The ex-post stability of f implies $f_{i x_1}(R') = 1$ for all $i \in N_1 \setminus N_1^*$. Hence, the allocation of x_1 remains the same at R' and $f_{i_1 x_1}(R') = 0$. Let μ be any stable deterministic allocation for $(\succeq, R')$. If $\mu(i_1) = \emptyset$, then $\mu(i_2) \neq x_1$ and $\mu(i_3) \neq x_1$. This implies that x_1 is wasted. So by ex-post stability, $f_{i_1 \emptyset}(R') = 0$. Therefore, $f_{i_1 x_{n-1}}(R') = 1$. Consider any $i \in N_{n-1}$. If $i \succ_{x_{n-1}} i_1$, then by ex-post stability $f_{i x_{n-1}}(R') = 1$. If $i \sim_{x_{n-1}} i_1$, then by symmetry at the top $f_{i x_{n-1}}(R') = 1$. It follows that $f_{i_n x_{n-1}}(R') = 0$. Since $i_n \succeq_{x_{n-1}} i_1$, by ex-post stability (when $i_n \succ_{x_{n-1}} i_1$) or local envy-freeness (when $i_n \sim_{x_{n-1}} i_1$), we have $f_{i_n x_{n-2}}(R') = 1$. By similar arguments, it can be shown that $f_{i_{n-1} x_{n-3}}(R') = 1$ (if $n \geq 4$) and so on. Eventually this leads to $f_{i_3 x_1}(R') = 1$. Then it follows from previous discussion that $f_{i x_1}(R') = 1$ for all $i \in \{i_2, i_3\} \cup N_1$, which is impossible.

Case 2: There is a type-II* cycle.

Let this cycle consist of $i, j, k \in N$ and $x, y \in X$ such that: (1) $i \sim_x j \sim_x k \succ_y i$ and $j \succ_y i$; (2) there exist disjoint $N_x, N_y \subseteq N \setminus \{i, j, k\}$ such that $N_x \subseteq U(\succeq_x, i)$, $N_x \subseteq SU(\succeq_y, i)$, $N_y \subseteq SU(\succeq_y, i)$, $|N_x| = q_x - 1$ and $|N_y| = q_y - 1$. Similar to Case 2 in the proof of $(iii) \Rightarrow (v)$, let $N_x^* = \{l \in N_x : l \sim_x i\}$. Then in the following analysis, each

$l \in N_x \setminus N_x^*$ will always be assigned x with probability one. Let $q = q_x - |N_x \setminus N_x^*|$, then $|N_x^*| = q - 1 \geq 0$. Consider the following preference profiles.

$$
R: \begin{array}{ll}
i: & y, x, \emptyset \\
j: & x, \emptyset \\
k: & x, \emptyset \\
N_x^*: & x, \emptyset \\
N_y: & y, \emptyset \\
N_x \setminus N_x^*: & x, \emptyset
\end{array}
\qquad
R': \begin{array}{ll}
i: & y, x, \emptyset \\
j: & x, y, \emptyset \\
k: & x, y, \emptyset \\
N_x^*: & x, y, \emptyset \\
N_y: & y, \emptyset \\
N_x \setminus N_x^*: & x, \emptyset
\end{array}
$$

By individual rationality and ex-post nonwastefulness, we have $f_{iy}(R) = 1$, $f_{ix}(R) = 0$ and $\sum_{l \in \{j,k\} \cup N_x^*} f_{lx}(R) = q$. By symmetry at the top, $f_{lx}(R) = \frac{q}{q+1}$ for all $l \in \{j,k\} \cup N_x^*$. By strategy-proofness, $f_{jx}(R'_j, R_{-j}) = \frac{q}{q+1}$. By symmetry at the top, $f_{lx}(R'_j, R_{-j}) = \frac{q}{q+1}$ for all $l \in \{j,k\} \cup N_x^*$. We can use the same reasoning to show $f_{lx}(R'_{\{j,k\}}, R_{-\{j,k\}}) = \frac{q}{q+1}$ for all $l \in \{j,k\} \cup N_x^*$, and so on. In the end, we have $f_{lx}(R') = \frac{q}{q+1}$ for all $l \in \{j,k\} \cup N_x^*$. It follows that $f_{ix}(R') = 0$. Next, let μ be any stable deterministic allocation for $(\succeq, R')$. If $\mu(i) = y$, $\mu(l) = y$ for all $l \in N_y$. Then respecting priorities further requires $\mu(l) = x$ for all $l \in \{j,k\} \cup N_x^*$, which is not possible. So by ex-post stability, $f_{iy}(R') = 0$. Given that $f_{ix}(R') = f_{iy}(R') = 0$, i can manipulate by reporting $R''_i : x, \emptyset$, since by symmetry at the top $f_{ix}(R''_i, R'_{-i}) > 0$. This contradicts to the strategy-proofness of f. □

Proof of Theorem 2

It is already known that $(iii) \Rightarrow (i)$ and $(i) \Rightarrow (ii)$, we show $(ii) \Rightarrow (iii)$. Assume to the contrary, there exists a strategy-proof, ex-post stable-and-efficient, symmetric and locally envy-free rule f, but $\succeq$ is not strongly T-acyclic. Then by Theorem 1, $\succeq$ is not strongly acyclic. That is, there exists a weak cycle that consists of distinct $i, j, k \in N$ and $x, y \in X$ such that: (1) $i \succeq_x j \succ_x k \succeq_y i$; (2) there exist disjoint $N_x, N_y \subseteq N \setminus \{i, j, k\}$ such that $N_x \subseteq U(\succeq_x, j)$, $N_y \subseteq U(\succeq_y, i)$, $|N_x| = q_x - 1$ and $|N_y| = q_y - 1$. Consider the following preference profiles.

$$
R: \begin{array}{ll}
i: & y, x, \emptyset \\
j: & x, \emptyset \\
k: & x, y, \emptyset \\
N_x: & x, \emptyset \\
N_y: & y, \emptyset
\end{array}
\qquad
R': \begin{array}{ll}
i: & y, \emptyset \\
j: & x, \emptyset \\
k: & x, y, \emptyset \\
N_x: & x, \emptyset \\
N_y: & y, \emptyset
\end{array}
$$

Let μ be stable and efficient for $(\succeq, R)$. Suppose $\mu(i) = x$. Then by individual rationality and nonwastefulness $\mu(k) = y$, which contradicts efficiency. So by ex-post stability-and-efficiency, $f_{ix}(R) = 0$.

Next, we show $f_{iy}(R) < 1$. Suppose that $f_{iy}(R) = 1$. Then by strategy-proofness $f_{iy}(R') = 1$. Consider any $l \in N_y$. If $l \succ_y i$, then by ex-post stability $f_{ly}(R') = 1$. If $l \sim_y i$, then by symmetry $f_{ly}(R') = 1$. Therefore, $f_{ky}(R') = 0$. As $k \succeq_y i$, by ex-post stability (when $k \succ_y i$) or local envy-freeness (when $k \sim_y i$), $f_{kx}(R') = 1$. Since $(N_x \cup \{j\}) \subseteq SU(\succeq_x, k)$, the ex-post stability of f implies $f_{lx}(R') = 1$ for all $l \in N_x \cup \{j\}$, which is clearly impossible.

Given that $f_{ix}(R) = 0$ and $f_{iy}(R) < 1$, we have $f_{i\emptyset}(R) > 0$. Then by ex-post stability (when $i \succ_x j$) or local envy-freeness (when $i \sim_x j$), $f_{jx}(R) < 1$. Let λ be a lottery that induces $f(R)$ such that each $\mu \in \mathscr{S}(\lambda)$ is stable and efficient. Then there exists $\mu \in \mathscr{S}(\lambda)$ with $\mu(j) = \emptyset$ and $\mu(i) \neq x$. By stability, $\mu(k) \neq x$. Therefore, by individual rationality x is assigned to at most $|N_x|$ agents at μ, contradicting to the nonwastefulness of μ. □

Appendix B: Independence of Axioms

Theorem 1 and Theorem 2 include the following three impossibility results:

I.1. *If $\succeq$ is not T-acyclic, there does not exist a strategy-proof, ex-post constrained efficient, symmetric and locally envy-free rule.*
I.2. *If $\succeq$ is not T-acyclic, there does not exist a strategy-proof, ex-post stable, symmetric at the top and locally envy-free rule.*
I.3. *If $\succeq$ is not strongly T-acyclic, there does not exist a strategy-proof, ex-post stable-and-efficient, symmetric and locally envy-free rule.*

In this section we briefly discuss the independence of axioms in these results. We keep the following assumption, $A.1$, throughout this section. If $A.1$ is not satisfied, $\succeq$ is guaranteed to be strongly T-acyclic.

A.1. *There exist distinct $x, y \in X$ such that $|N| \geq q_x + q_y + 1$.*

Stronger impossibility results can be obtained in the special case of one-to-one allocation problems. In fact, those stronger results can be established as long as the objects in a cycle have unit capacities.

A.2. *There exist distinct $x, y \in X$ such that $|N| \geq q_x + q_y + 1$ and $q_x \geq 2$.*

It can be easily seen that, when $A.2$ is not satisfied, a type-I, type-II or weak cycle can only involve objects with unit capacities. In this case, symmetry can be dropped from $I.1$, and both symmetry and strategy-proofness can be dropped from $I.3$.

I.1*. *Suppose that A.2 is not satisfied. If $\succeq$ is not T-acyclic, there does not exist a strategy-proof, ex-post constrained efficient and locally envy-free rule.*

Proof. Suppose that $A.2$ is not satisfied and $\succeq$ is not T-acyclic.

First, consider the case in which there exists a type-I cycle. We use arguments similar to those in Case 1, proof of $(iii) \Rightarrow (v)$, Theorem 1, to show that there does not exist an ex-post constrained efficient and locally envy-free rule. Assume to the contrary, there exists such a rule f. Let the type-I cycle consist of distinct $i_1, \ldots, i_n \in N$, $n \geq 3$, and distinct $x_1, \ldots, x_{n-1} \in X$ such that $q_{x_k} = 1$ for all k with $1 \leq k \leq n-1$, $i_1 \succ_{x_1} i_2 \sim_{x_1} i_3$, and $i_{k+1} \succeq_{x_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$. Consider the following preference profile.

$$
\begin{array}{lll}
 & i_1: & x_{n-1}, x_1, \emptyset \\
R: & i_2: & x_1, \emptyset \\
 & i_k: & x_{k-2}, x_{k-1}, \emptyset \quad \text{for } k = 3, \ldots, n
\end{array}
$$

Let μ be a deterministic allocation for $(\succeq, R)$. If $\mu(i_1) = x_1$, then μ is Pareto dominated by the stable allocation ν in which $\nu(i_2) = \emptyset$ and every other agent is assigned her top choice. So by ex-post constrained efficiency, $f_{i_1 x_1}(R) = 0$. Next, suppose $f_{i_1 x_{n-1}}(R) = 1$. Then $f_{i_n x_{n-1}}(R) = 0$. By ex-post stability (when $i_n \succ_{x_{n-1}} i_1$) or local envy-freeness (when $i_n \sim_{x_{n-1}} i_1$), $f_{i_n x_{n-2}}(R) = 1$. Continuing in this fashion, in the end we have $f_{i_3 x_1}(R) = 1$. Then $f_{i_2 \emptyset}(R) = 1$, contradicting to local envy-freeness. Finally, given that $f_{i_1 x_1}(R) = 0$ and $f_{i_1 x_{n-1}}(R) < 1$, $f_{i_1 \emptyset}(R) > 0$. Let λ be a lottery that induces $f(R)$ such that each $\mu' \in \mathscr{S}(\lambda)$ is constrained efficient. There exists $\mu' \in \mathscr{S}(\lambda)$ such that $\mu'(i_1) = \emptyset$. Then by stability, $\mu'(i_2) \neq x_1$ and $\mu'(i_3) \neq x_1$. Therefore, a contradiction is reached since x_1 is wasted.

Second, consider the case in which there exists a type-II* cycle. So there exist distinct $i, j, k \in N$ and $x, y \in X$ such that $q_x = q_y = 1$, $i \sim_x j \sim_x k \succ_y i$ and $j \succ_y i$. Without loss of generality, let $k \succeq_y j$. To obtain a contradiction, assume that there exists a strategy-proof, ex-post constrained efficient and locally envy-free rule f. Consider the following preference profiles.

$$R^1: \begin{array}{ll} i: & y, x, \emptyset \\ j: & y, x, \emptyset \\ k: & x, y, \emptyset \end{array} \qquad R^2: \begin{array}{ll} i: & y, x, \emptyset \\ j: & x, y, \emptyset \\ k: & x, y, \emptyset \end{array} \qquad R^3: \begin{array}{ll} i: & y, x, \emptyset \\ j: & x, \emptyset \\ k: & x, \emptyset \end{array}$$

Consider any constrained efficient allocation μ for $(\succeq, R^1)$. Suppose that $\mu(j) = x$. Then by stability $\mu(k) = y$. But μ is Pareto dominated by the stable allocation ν in which $\nu(j) = y$ and $\nu(k) = x$, contradiction. So by ex-post constrained efficiency, $f_{jx}(R^1) = 0$. Next, suppose that $f_{jy}(R^1) = 1$. Then $f_{ky}(R^1) = 0$. By ex-post stability (when $k \succ_y j$) or local envy-freeness (when $k \sim_y j$), $f_{kx}(R^1) = 1$. Then $f_{i\emptyset}(R^1) = 1$, contradicting to local envy-freeness. Therefore, $f_{jy}(R^1) + f_{jx}(R^1) < 1$.

By strategy-proofness, $f_{jy}(R^2) + f_{jx}(R^2) < 1$. Then by ex-post nonwastefulness, $f_{iy}(R^2) + f_{ix}(R^2) > 0$. It can be easily seen that for any stable deterministic allocation μ for $(\succeq, R^2)$, $\mu(i) \neq y$. So by ex-post stability $f_{iy}(R^2) = 0$. Therefore, $f_{ix}(R^2) > 0$.

Let μ be any constrained efficient allocation for $(\succeq, (R^3_j, R^2_{-j}))$. Suppose that $\mu(i) = x$. Then by stability $\mu(k) = y$. But μ is Pareto dominated by the stable allocation ν in which $\nu(i) = y$ and $\nu(k) = x$, contradiction. So by ex-post constrained efficiency, $f_{ix}(R^3_j, R^2_{-j}) = 0$. Similarly, it can be shown that $f_{ix}(R^3_k, R^2_{-k}) = 0$.

By strategy-proofness, $f_{jx}(R^3_j, R^2_{-j}) = f_{jx}(R^2)$. Then $f_{ix}(R^2) > 0$ and $f_{ix}(R^3_j, R^2_{-j}) = 0$, together with the ex-post nonwastefulness of f, imply $f_{kx}(R^3_j, R^2_{-j}) = f_{kx}(R^2) + f_{ix}(R^2)$. Similarly, it can be shown that $f_{jx}(R^3_k, R^2_{-k}) = f_{jx}(R^2) + f_{ix}(R^2)$.

Finally, by strategy-proofness, $f_{jx}(R^3) = f_{jx}(R^3_k, R^2_{-k}) = f_{jx}(R^2) + f_{ix}(R^2)$, and $f_{kx}(R^3) = f_{kx}(R^3_j, R^2_{-j}) = f_{kx}(R^2) + f_{ix}(R^2)$. A contradiction is reached since $f_{jx}(R^3) + f_{kx}(R^3) = 2f_{ix}(R^2) + f_{jx}(R^2) + f_{kx}(R^2) > 1$. □

I.3*. *Suppose that A.2 is not satisfied. If $\succeq$ is not strongly T-acyclic, there does not exist an ex-post stable-and-efficient, and locally envy-free rule.*

Proof. Suppose that $A.2$ is not satisfied and $\succeq$ is not strongly T-acyclic, but there exists an ex-post stable-and-efficient, and locally envy-free rule f. Recall that in the proof of $I.1^*$, it is shown that an ex-post constrained efficient and locally envy-free rule does not exist if there is a type-I cycle. So there exists a type-II* cycle or a weak cycle. It can be easily seen that a type-II* cycle also involves a weak cycle. Hence, it is sufficient to consider the case in which there are distinct $i, j, k \in N$ and $x, y \in X$ such that $q_x = q_y = 1$ and $i \succeq_x j \succ_x k \succeq_y i$. Consider the following preference profile.

$$
R: \quad \begin{array}{ll} i: & y, x, \emptyset \\ j: & x, \emptyset \\ k: & x, y, \emptyset \end{array}
$$

Similar to the proof of Theorem 2, it is easy to show the following. First, by ex-post stability-and-efficiency, $f_{ix}(R) = 0$. Second, by ex-post stability or local envy-freeness, if $f_{iy}(R) = 1$, then $f_{kx}(R) = 1$. But this implies $f_{jx}(R) = 0$ and contradicts ex-post stability. Hence, $f_{iy}(R) < 1$.

Since $f_{i\emptyset}(R) > 0$, by ex-post stability (when $i \succ_x j$) or local envy-freeness (when $i \sim_x j$), $f_{jx}(R) \neq 1$. Let λ be a lottery that induces $f(R)$ such that each $\mu \in \mathscr{S}(\lambda)$ is stable and efficient. Then there exists $\mu \in \mathscr{S}(\lambda)$ with $\mu(j) = \emptyset$ and $\mu(i) \neq x$. As μ respects priorities, $\mu(k) \neq x$. So x is wasted and a contradiction is reached. $\square$

In the rest of this section, we give examples to show that the impossibility results can not be further strengthened.

Independence of the axioms in I.1. First, the following example shows that local envy-freeness is crucial (for all the impossibility results).

Example B.1 ($\succeq$ *is not T-acyclic, and there exists a strategy-proof, ex-post stable-and-efficient, and strongly symmetric rule*). By *A*.1, there exist distinct $1, 2, 3 \in N$ and $x, y \in X$ such that $|N| \geq q_x + q_y + 1$. Let $\sigma \in \mathscr{O}$. The priority structure is as follows: $1 \sim_x 3 \succ_x 2$, $1 \succ_z 2 \sim_z 3$ for all $z \in X \setminus \{x\}$, $N \setminus \{1, 2, 3\} \subseteq SU(\succeq_z, 1)$ for all $z \in X$, and for all $z \in X$ and $i, j \in N \setminus \{1, 2, 3\}$, $i \succ_z j$ whenever $\sigma(i) < \sigma(j)$. $\succeq$ is not T-acyclic since there is a type-I cycle in which $1 \succ_y 2 \sim_y 3$ and $3 \succeq_x 1$. Construct a rule f as follows. First, the agents in $N \setminus \{1, 2, 3\}$ choose their best available choices sequentially according to the ordering σ, and leave the problem with their assignments. Then, consider the reduced problem.

(1) x is exhausted or there are at least two copies of x available. Then the priority structure in the reduced problem is strongly T-acyclic. So apply DA-STB to it.

(2) There is exactly one copy of x available.

 (a) Agent 1's first choice is $z \in \bar{X} \setminus \{x\}$. 1 is assigned z and leaves the problem with her assignment. Apply DA-STB to the further reduced problem.

 (b) Agent 1's first choice is x.

  (i) Agent 3's first choice is x. Each of 1 and 3 is assigned 0.5 of x. After x is exhausted, the three agents consume the probability shares of their best available choices sequentially in the order of $1, 2, 3$.[38]

(ii) Agent 3's first choice is not x. 1 is assigned x and leaves the problem with x. Apply DA-STB to the further reduced problem.

It is not difficult to verify that f is strategy-proof, ex-post stable-and-efficient, and strongly symmetric. The key in this construction is that (i) and (ii) guarantee that agent 3 cannot manipulate. Notice that in case (b) agent 3 can only get a share of x by reporting it as her first choice, but then she has to consume another object after agent 2. □

Second, strategy-proofness cannot be dropped.

Example B.2 ($\succeq$ *is not T-acyclic, and there exists an ex-post constrained efficient, symmetric and locally envy-free rule*). The setup is the same as Example B.1 except the relative priority rankings among 1,2 and 3: let $1 \sim_x 2 \sim_x 3$ and $1 \succ_z 2 \succ_z 3$ for all $z \in X \setminus \{x\}$. Then $\succeq$ is not T-acyclic. Construct a rule f as follows. First, as in the previous example, let the agents in $N \setminus \{1, 2, 3\}$ choose their best available choices sequentially according to σ, and leave the problem with their assignments. Then, consider the reduced problem.

- x is exhausted or there are at least two copies of x available. Then the priority structure in the reduced problem is T-acyclic. So apply DA-STB to it.
- There is exactly one copy of x available.
  (1) Agent 1's first choice is $z \in \bar{X} \setminus \{x\}$. 1 is assigned z and leaves the problem with her assignment. Apply DA-STB to the further reduced problem.
  (2) Agent 1's first choice is x, and there are in total $n > 1$ agents whose first choices are x. Then each of these n agents is assigned $\frac{1}{n}$ of x. After x is exhausted, the three agents consume the probability shares of their best available choices sequentially in the order of $1, 2, 3$.
  (3) Agent 1 is the only agent whose first choice is x.
    (a) Agent 2 and agent 3 do not have the same first choice, or their common first choice has more than one copies available. Then each agent is

[38] For example, their preferences are as follows. $R_1 : x, y$; $R_2 : x, z$; $R_3 : x, z, y$. Each object has unit capacity. Then agent 1 consumes 0.5 of y, agent 2 consumes 1 of z and agent 3 consumes 0.5 of y. This is essentially an *eating algorithm* from Bogomolnaia and Moulin (2001). Hence in this case the resulting allocation M is *sd-efficient*, i.e., there does not exist a different allocation M' such that M'_i first-order stochastically dominates M_i for all $i \in N$. It can also be verified that M is *ex-ante stable* (Roth et al., 1993), which implies that there do not exist $i, j \in N$ and $z \in X$ such that $i \succ_z j$, $M_{jz} > 0$ and $\sum_{o \in U(R_i, z)} M_{io} < 1$. It follows from sd-efficiency and ex-ante stability that M must be ex-post stable-and-efficient.

assigned her first choice.

(b) Agent 2 and agent 3 have the same first choice with exactly one copy available.

(i) Agent 3's second choice is not x. Then 1 and 2 are assigned their first choices, and 3 is assigned her second choice.

(ii) Agent 3's second choice is x. Then each of 1 and 3 is assigned 0.5 of x. After x is exhausted, the three agents consume their best available choices sequentially in the order of $1, 2, 3$.

The symmetry and local envy-freeness of f can be easily verified. f is ex-post stable-and-efficient except one case in (ii): if 1's second choice coincides with 3's first choice, then the allocation chosen by f is only ex-post constrained efficient. $\square$

Third, under the assumption $A.2$, symmetry cannot be dropped. In the following example, we additionally show that strategy-proofness cannot be dropped from $I.3$ when $A.2$ is satisfied.

Example B.3 ($A.2$ *is satisfied and* $\succeq$ *is not T-acyclic. There exists a strategy-proof, ex-post stable-and-efficient, and locally envy-free rule, which also satisfies equal treatment of equals, and there exists an ex-post stable-and-efficient, locally envy-free and symmetric rule*). By $A.2$, we can find distinct $x, y \in X$ such that $q_x \geq 2, q_x \leq q_z$ for all $z \in X$ with $q_z \geq 2$, $q_y \leq q_z$ for all $z \in A \setminus \{x\}$, and $|N| \geq q_x + q_y + 1$. Let $1 \in N$ and $N' \subseteq N$ such that $1 \notin N'$ and $|N'| = q_x + q_y$. Consider the following priority structure. For any $z \in X \setminus \{x\}$, $i, j \in N'$ and $k \in N \setminus (N' \cup \{1\})$, we have $1 \sim_x i \sim_x j \succ_x k$ and $1 \succ_z i \sim_z j \succ_z k$. Moreover, given $\hat{\sigma} \in \mathcal{O}$, let $i \succ_z j$ if $z \in X$, $i, j \in N \setminus (N' \cup \{1\})$ and $\hat{\sigma}(i) < \hat{\sigma}(j)$. It can be easily seen that $\succeq$ is not T-acyclic. Construct a rule f as follows. If agent 1's first choice is not x, let 1 be assigned her first choice and leave with her assignment. Then apply DA-STB to the reduced problem. If agent 1's first choice is x, apply a modified random serial dictatorship in which each ordering $\sigma \in \mathcal{O}$ satisfying the following conditions is picked with equal probability: $\sigma(1) \leq |N'|, \sigma(i) \leq |N'| + 1$ for all $i \in N'$, and $\sigma(i) < \sigma(j)$ for all $i, j \in N \setminus (N' \cup \{1\})$ with $\hat{\sigma}(i) < \hat{\sigma}(j)$. Denote the set of such orderings as $\mathcal{O}'$.

First, note that any two agents with the same priority at each object are in the set N'. Then f satisfies equal treatment of equals by the construction. Next, let $f^{SD}(\sigma, \cdot)$ be the serial dictatorship rule with respect to an ordering σ. For any R and $\sigma \in \mathcal{O}'$, if agent 1's first two choices are x and $z \in \bar{X} \setminus \{x\}$, then $f_1^{SD}(\sigma, R) R_1 z$, since $q_x + q_z \geq |N'| \geq \sigma(1)$.

Given this, it can be easily seen that f is strategy-proof, and ex-post stable-and-efficient. It remains to show local envy-freeness. Suppose that for some R and $i \in N'$, $f_{1x}(R) = 1$, $f_{ix}(R) = 0$ and $f_{iz}(R) > 0$ for some $z \in \bar{X}$ with $x P_i z$. Then there exists $\sigma \in \mathcal{O}'$ such that $f_1^{SD}(\sigma, R) = x$ and $f_i^{SD}(\sigma, R) = z$. As $q_x \geq 2$, there is some $j \in N'$ with $f_j^{SD}(\sigma, R) = x$. Clearly $\sigma(1) < \sigma(i)$ and $\sigma(j) < \sigma(i)$. Construct two new orderings based on σ: let σ' be the ordering obtained by moving 1 down such that 1 is just above i;[39] let σ'' be the ordering obtained (from σ) by moving j down such that j is just below i. Then $\sigma', \sigma'' \in \mathcal{O}'$. First, we argue that $f_k^{SD}(\sigma, R) R_k x$ for all $k \in N$ with $\sigma(k) < \sigma(i)$. Assume this is not true. Then let $k \neq i$ be the first agent whose assignment is worse than x. Clearly $\sigma(1) < \sigma(k) < \sigma(i)$. Consider any agent l above k with $l \neq 1$. Since l chooses an object weakly better than x under σ, l must choose the same object under σ'. It follows that $f_k^{SD}(\sigma', R) = x$ and $f_1^{SD}(\sigma', R) \neq x$, contradicting to $f_{1x}(R) = 1$. Next, consider any agent l above i with $l \neq j$. Since l chooses an object weakly better than x under σ, l chooses the same object under σ''. This implies $f_i^{SD}(\sigma'', R) = x$, contradicting to $f_{ix}(R) = 0$. Local envy-freeness with respect to two agents in N' can be shown in a similar (and easier) way.

We construct another rule f' as follows.

(1) Agent 1's first choice is not x, or, her first choice is x and her second choice is not $\emptyset$. Let $f' = f$.

(2) Agent 1's first choice is x and her second choice is $\emptyset$. Apply a modified random serial dictatorship in which each ordering $\sigma \in \mathcal{O}$ satisfying the following conditions is picked with equal probability: $\sigma(i) \leq |N'| + 1$ for all $i \in N' \cup \{1\}$, and $\sigma(i) < \sigma(j)$ for all $i, j \in N \setminus (N' \cup \{1\})$ with $\hat{\sigma}(i) < \hat{\sigma}(j)$.

It can be easily checked that f' is ex-post stable-and-efficient, locally envy-free and symmetric. Notice that in case (1) the symmetry constraints only apply to those agents in N', and in case (2) agent 1's priority cannot be violated by any agent in N'. $\square$

Finally, in $I.1$ ex-post constrained efficiency cannot be weakened to ex-post stability, as DA-STB is always strategy-proof, ex-post stable, symmetric and locally envy-free.

Independence of the axioms in I.2. The *fractional deferred acceptance mechanism* from Kesten and Ünver (2015) is ex-post stable, symmetric at the top and locally envy-free. Random serial dictatorship is strategy-proof, symmetric at the top and locally

[39] Formally, $\sigma'(1) = \sigma(i) - 1$, $\sigma'(k) = \sigma(k) - 1$ if $\sigma(1) < \sigma(k) < \sigma(i)$, and $\sigma'(k) = \sigma(k)$ if $\sigma(k) < \sigma(1)$ or $\sigma(k) \geq \sigma(i)$.

envy-free. DA-STB is strategy-proof, ex-post stable and locally envy-free. Finally, local envy-freeness cannot be dropped, as shown in Example B.1.

Independence of the axioms in I.3. DA-STB is strategy-proof, ex-post stable, symmetric and locally envy-free. Random serial dictatorship is strategy-proof, ex-post efficient, symmetric and locally envy-free. Local envy-freeness cannot be dropped as shown in Example B.1. Finally, when $A.2$ is satisfied, strategy-proofness or symmetry cannot be dropped as shown in Example B.3.

Appendix C

The following example from Han (2015) shows that an ex-post stable and ex-post efficient allocation may not be ex-post stable-and-efficient.

Example C.1 (Han, 2015). Suppose that $N = \{1,2,3,4,5\}$, $X = \left\{x_1, x_2, x_3, x_4, x_5\right\}$ and all the objects have unit-capacities. The priority structure is as follows: $i \sim_x j$ for all $i, j \in N \setminus \{5\}$ and $x \in X$; $i \succ_x 5$ for all $i \in N \setminus \{5\}$ and $x \in X$. Consider the following preference profile and allocation M, where an underlined object is assigned to the corresponding agent with probability 0.5.

$$\begin{array}{rc} 1: & \underline{x_1}, \underline{x_2}, \emptyset \\ 2: & \underline{x_1}, x_5, \underline{x_2}, \emptyset \\ 3: & \underline{x_4}, \underline{x_3}, \emptyset \\ 4: & x_2, \underline{x_4}, \underline{x_5}, \emptyset \\ 5: & \underline{x_5}, \underline{x_3}, \emptyset \end{array}$$

For simplicity, we use $\mu^1 = (x_1, x_2, x_3, x_4, x_5)$ to denote the deterministic allocation μ^1 in which $\mu^1(1) = x_1, \mu^1(2) = x_2, \mu^1(3) = x_3, \mu^1(4) = x_4$ and $\mu^1(5) = x_5$. Let $\mu^2 = (x_2, x_1, x_4, x_5, x_3)$, $\mu^3 = (x_1, x_2, x_4, x_5, x_3)$ and $\mu^4 = (x_2, x_1, x_3, x_4, x_5)$. Then both the lottery λ, in which $\lambda(\mu^1) = \lambda(\mu^2) = \frac{1}{2}$, and the lottery λ', in which $\lambda'(\mu^3) = \lambda'(\mu^4) = \frac{1}{2}$, induce M. It can be easily verified that μ^1 and μ^2 are efficient, and μ^3 and μ^4 are stable. Therefore, M is ex-post stable and ex-post efficient. However, it is not ex-post stable-and-efficient. To see this, let λ'' be any lottery that induces M. There exists $\mu \in \mathscr{S}(\lambda'')$ such that $\mu(2) = x_2$. Suppose that μ is efficient. Then $\mu(4) = x_4$. Since $M_{3x_4} = M_{3x_3} = M_{5x_3} = M_{5x_5} = \frac{1}{2}$, we have $\mu(3) = x_3$ and $\mu(5) = x_5$. But $x_5 P_2 x_2$ and

$2 \succ_{x_5} 5$, so μ is not stable. Therefore, some deterministic allocation in the support of λ'' cannot be both stable and efficient.

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