Characterizing Priorities for Deferred Acceptance With or Without Outside Options

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Abstract

In a model of priority-based allocation of indivisible objects where there may not be outside options, we characterize the priority structures under which the deferred acceptance algorithm (DA) satisfies various desiderata. We first identify an acyclicity condition that is necessary and sufficient for DA to be group strategy-proof, robustly stable, weakly group robustly stable, or to implement the stable allocation correspondence in Nash equilibria. When there is no outside option and there are more agents than total resources, the condition becomes considerably weaker, and no longer requires the priorities between any pair of objects to be similar. We further find a condition on priorities that is necessary and sufficient for the efficiency or consistency of DA, which is in general stronger than the above incentive properties for this mechanism.

Keywords: priority-based allocation; outside option; deferred acceptance algorithm; group strategy-proofness; efficiency; acyclicity

JEL Codes: C78; D47; D78; D82

1 Introduction

Many indivisible object allocation problems, such as *school choice* (Abdulkadiroğlu and Sönmez, 2003), do not involve monetary transfers and are priority-based, where each

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object has a strict priority ordering over agents, and each agent has a strict preference relation over the objects as well as her outside option. The *deferred acceptance algorithm* (DA) from Gale and Shapley (1962) is the most prominent mechanism for such problems. While it is strategy-proof (Dubins and Freedman, 1981, Roth, 1982) and agent-optimal stable, DA fails to satisfy several other desirable properties including Pareto efficiency. It is well-known that this is attributed to *rejection cycles* in its apply-and-reject procedure, where an agent's application to an object induces a sequence of rejections that makes herself rejected by the object in the end. Ergin (2002) introduces a condition on the priority structure, which we refer to as *E-acyclicity*, that rules out such rejection cycles, and turns out to be necessary and sufficient for DA to be efficient, group strategy-proof, or consistent. Following studies provide further evidence on the key role of this condition in ensuring other properties of DA, as we detail later.

In this paper, we restrict preferences in the above classical model by assuming all objects to be acceptable for each agent. A simple and direct interpretation is that there is no outside option and receiving any object is better than receiving nothing, which naturally models some real-life scenarios. However, outside options are also permitted and the classical model is included as a special case: in our many-to-one setting we can simply let an object with a sufficiently large capacity assume the role of outside option for each agent.[1]

In this context, a key first observation is that E-acyclicity is generally not necessary for the group strategy-proofness of DA, due to the fact that some "persistent" rejection cycles, which are ruled out by E-acyclicity, do not create chances for joint manipulations. For instance, when an agent's application to some object induces a rejection cycle and the agent eventually receives nothing, she may not be able to misreport her preferences to prevent the rejection cycle and help improve the assignments of others. This is in contrast to the classical model, where the agent can always opt out by reporting her outside option as the top choice.

We then identify a weaker condition that is necessary and sufficient for DA to be group strategy-proof. E-acyclicity is defined as the absence of *E-cycles* in priority relations, and our condition of *acyclicity* is defined by ruling out some particular E-cycles that are embedded in *stable allotments*. A stable allotment consists of a group of agents and their allotted objects so that such allotments do not violate other agents' priorities

[1]Recently Kesten and Kurino (2019) take a similar approach and study strategy-proof improvements over DA in a model where there may or may not be outside options. Their setup is even more general as agent-specific outside options are allowed.

for all possible preferences. In the first theorem, we show that acyclicity characterizes the priority structures under which DA is group strategy-proof, as well as the ones under which (1) DA is *robustly stable* (Kojima, 2011), i.e., the mechanism is immune to any combined manipulation where an agent first misreports her preferences and then blocks the allocation, (2) DA satisfies *weak group robust stability* (Afacan, 2012), which strengthens robust stability by eliminating combined manipulations by coalitions, or (3) DA implements the stable allocation correspondence in Nash equilibria.

It has been shown by Ergin (2002), Kojima (2011), Afacan (2012) and Haeringer and Klijn (2009) respectively that in the classical model each of the above incentive properties is satisfied by DA if and only if the priority structure is E-acyclic. Our theorem first implies these results, since acyclicity is reduced to E-acyclicity when the number of agents is less than or equal to the total capacities of objects, which covers the case that some object with a large capacity plays the role of outside option in our model.

On the other hand, as long as the number of agents is greater, these two conditions diverge and acyclicity is considerably weaker with different interpretations. It is well-known that E-acyclicity is very restrictive, and essentially requires any two objects to rank the agents besides a top group similarly. In contrast, when there are more agents, we show that acyclicity is satisfied if any two objects rank the agents besides a top group either similarly, or very differently. In face of object shortages, heterogenous priority orderings between objects help prevent the formation of a stable allotment that includes agents in an E-cycle and make possible rejection cycles persistent, reducing the incentives to manipulate.

In the end, we study three additional axioms. It turns out that efficiency is in general stronger than group strategy-proofness for DA. We thus introduce *strong acyclicity*, and show that it is necessary and sufficient for DA to be efficient, or consistent. It is defined as the absence of an E-cycle embedded into a stable allotment in a different way, and rules out rejection cycles in the DA procedure. Furthermore, in the (extended) classical model Kesten (2012) shows that each object has no incentive to under-report its capacity if and only if the priority structure with respect to minimum possible capacities is E-acyclic. This still holds in our model, due to the flexible capacities under consideration.

In the next section we present the model and some basic existing results. Section 3 gives our main results on the incentive properties for agents, while Section 4 considers the three additional axioms. Section 5 concludes. All the proofs are in the appendix.

2 Preliminaries

2.1 The Model

Let N be a finite set of **agents**, and A a finite set of **objects**. Assume $|A| \geq 2$. For each object $a \in A$, there are $q_a \geq 1$ copies, and a has a complete, transitive and antisymmetric **priority ordering** $\succeq_a$ on N, with $\succ_a$ denoting its asymmetric component. Given $i \in N$, let $U(\succ_a, i) = \{j \in N : j \succ_a i\}$. A **priority structure** $\succeq = (\succeq_a)_{a \in A}$ is a profile of priority orderings. Let $\emptyset$ denote the null object (i.e., the outcome of being unassigned) with a capacity of $q_\emptyset = |N|$, and $\bar{A} = A \cup \{\emptyset\}$. Each agent $i \in N$ has a complete, transitive and antisymmetric **preference relation** R_i on $\bar{A}$, with P_i denoting its asymmetric component. As the central assumption in this study, we assume that $aP_i\emptyset$ for all $a \in A$ and $i \in N$. A **preference profile** $R = (R_i)_{i \in N}$ is a list of individual preferences. We fix N, A, $(q_a)_{a \in A}$ and $\succeq$. Then a **priority-based allocation problem**, or simply a **problem**, is represented by a preference profile R.

An **allocation** is denoted by a function $\mu : N \to \bar{A}$, where $|\mu^{-1}(a)| \leq q_a$ for all $a \in \bar{A}$. Given R, an allocation μ **Pareto dominates** another allocation ν if $\mu(i)R_i\nu(i)$ for all $i \in N$ and $\mu(j)P_j\nu(j)$ for some $j \in N$. An allocation is **efficient** if it can not be Pareto dominated by any allocation. An allocation μ is **non-wasteful** if $|\mu^{-1}(a)| < q_a$ implies $\mu(i)R_ia$ for all $i \in N$ and $a \in A$. It is **stable** if it is non-wasteful and there do not exist $i, j \in N$ such that $\mu(j)P_i\mu(i)$ and $i \succ_{\mu(j)} j$.

A **mechanism** is a function that assigns an allocation to each problem. A mechanism f is said to satisfy a certain property defined above if $f(R)$ satisfies this property for all R. f is **strategy-proof** if for every R, $i \in N$ and R'_i, $f_i(R)R_if_i(R'_i, R_{-i})$. It is **non-bossy** if for every R, $i \in N$ and R'_i, $f_i(R) = f_i(R'_i, R_{-i})$ implies $f(R) = f(R'_i, R_{-i})$. It is **group strategy-proof** if there do not exist R, $I \subseteq N$, and R'_I such that $f_i(R'_I, R_{-I})R_if_i(R)$ for all $i \in I$ and $f_j(R'_I, R_{-I})P_jf_j(R)$ for some $j \in I$. It is straightforward to check that the following well-known result still holds in our model:

Lemma 1 (Pápai, 2000). *f is group strategy-proof if and only if f is strategy-proof and non-bossy.*

2.2 Deferred Acceptance Algorithm

For any problem R, the **deferred acceptance algorithm** (DA) of Gale and Shapley (1962) selects an allocation through the following procedure:

Step 1. Each agent applies to her favorite object. Each object $a \in A$ places the applicants with the highest priorities up to its capacity q_a on its waiting list, and rejects the other applicants.

Step $k \geq 2$. Each agent who was rejected in Step $k-1$ applies to her next best object. Each object $a \in A$ chooses among the new applicants and the agents on its waiting list, places the ones with the highest priorities up to its capacity q_a on its waiting list, and rejects the others.

The procedure terminates when there are no more rejections. Then the copies of each object are assigned to the agents on its waiting list.

The outcome is stable, and Pareto dominates any other stable allocation. Moreover, the DA mechanism, denoted as f^{DA}, is strategy-proof.

In the *classical model* of priority-based allocation, the null object $\emptyset$, interpreted as the outside option, may be preferred to some objects by an agent. In this case, an acyclicity condition from Ergin (2002) has been shown to be necessary and sufficient for many desiderata of DA including efficiency and group strategy-proofness. For our purpose, we present the more general form of the cycles. An **E-cycle** consists of $n+1$ distinct agents $i_1, \ldots, i_n, i \in N$, where $n \geq 2$, and n distinct objects $a_1, \ldots, a_n \in A$ such that the following two conditions are satisfied:

- (Cycle) $i_1 \succ_{a_1} i \succ_{a_1} i_2$, and $i_k \succ_{a_k} i_{k+1}$ for all $k \in \{2, \ldots, n\}$, where $i_{n+1} = i_1$.
- (Scarcity) There are (possibly empty) mutually disjoint sets $N_1, \ldots, N_n \subseteq N \setminus \{i_1, \ldots, i_n, i\}$ such that $N_1 \subseteq U(\succ_{a_1}, i)$, $N_k \subseteq U(\succ_{a_k}, i_{k+1})$ for each $k \geq 2$, and $|N_k| = q_{a_k} - 1$ for each $k \in \{1, \ldots, n\}$.

Then the priority structure $\succeq$ is **E-acyclic** if there does not exist any E-cycle. As shown by Ergin (2002), when there is an E-cycle, there is an E-cycle with only three agents.

3 Results on Incentive Properties

We consider several properties of DA regarding the incentives of agents. While E-acyclicity is still sufficient for DA to be group strategy-proof, the following example illustrates that it is not necessary.

Example 1. *Suppose that $N = \{i, j, k\}$, $A = \{a, b\}$, and $q_a = q_b = 1$. The priorities and preferences are:*

$\succeq_a$	$\succeq_b$		R_i	R_j	R_k
i	k		b	a	a
j	i		a	b	b
k	j		$\emptyset$	$\emptyset$	$\emptyset$

The priority structure is not E-acyclic since $i \succ_a j \succ_a k \succ_b i$. This gives rises to a rejection cycle: in Step 1 of DA, k is rejected due to j's application to a; in Step 2, k displaces i at b; in Step 3, i displaces j at a. Consequently the DA outcome is not efficient, where $f_i^{DA}(R) = a$, $f_j^{DA}(R) = \emptyset$ and $f_k^{DA}(R) = b$. However, it can be checked that under such priority structure DA is group strategy-proof.

In the classical model, given the above true preferences R, j can report $\emptyset$ as her first choice so that the rejection cycle disappears, i and k receive their first choices, and j still receives $\emptyset$. Therefore, she is bossy and DA is not group strategy-proof. In contrast, in our model j always induces the rejection cycle regardless of her reported preferences.

For DA to be group strategy-proof, we no longer need to impose E-acyclicity and eliminate all possible rejection cycles. As the example shows, "persistent" rejection cycles do not create incentives for joint manipulations. We thus introduce a weaker condition that only eliminates the scenarios where an agent can control the appearance of a rejection cycle without affecting her own assignment. It is defined by ruling out some E-cycles that are embedded into the following structure.

Definition 1. *Given a non-empty set of agents $I \subseteq N$ and a function $s : I \to A$, if for every $i \in I$,*

$$\left| U(\succ_{s(i)}, i) \setminus \left\{ j \in I : s(j) \neq s(i) \right\} \right| < q_{s(i)},$$

*then we say (I, s) is a **stable allotment**.*

If (I, s) is a stable allotment, then assigning $s(i)$ to every agent $i \in I$ does not violate the priority of any agent in $N \setminus I$, regardless of the preferences of $N \setminus I$. Furthermore, given R, if $s(i)$ is the top choice for each $i \in I$, then $f_i^{DA}(R) = s(i)$ for each $i \in I$. On the other hand, if μ is a stable allocation for some R, and $I = \{i \in N : \mu(i) \neq \emptyset\}$ is the set of assigned agents, then $(I, \mu|_I)$ is a stable allotment.

Remark 1. The idea behind the construction of a stable allotment is essentially the same as *enforceability* in Rong, Tang, and Zhang (forthcoming). For non-empty $I \subseteq N$ and

$s: I \to \bar{A}$, they say the *subassignment* s is enforceable by the coalition I if the inequality in the above definition holds for every $i \in I$ with $s(i) \in A$.[2] Enforceability is further used to define a core concept in priority-based allocation, and there is no logical relation between their results and ours.

Definition 2. *A* ***cycle*** *consists of $n+1$ distinct agents $i_1, \ldots, i_n, i \in N$, where $n \geq 2$, such that the following conditions are satisfied:*

- *There exists a stable allotment (I, s) with $i_1, \ldots, i_n, i \in I$, and $s(i_1), \ldots, s(i_n), s(i)$ are distinct.*
- *(Cycle condition) $i_1 \succ_{s(i_2)} i \succ_{s(i_2)} i_2$, and $i_k \succ_{s(i_{k+1})} i_{k+1}$ for all $k \in \{2, \ldots, n\}$, where $i_{n+1} = i_1$.*
- *(Scarcity condition) There are (possibly empty) mutually disjoint sets $N_1, \ldots, N_n \subseteq I \setminus \{i_1, \ldots, i_n, i\}$ such that $N_2 \subseteq U(\succ_{s(i_2)}, i)$, and $N_k \subseteq U(\succ_{s(i_k)}, i_k)$ for $k \neq 2$. Moreover, for all $k \in \{1, \ldots, n\}$, $|N_k| = q_{s(i_k)} - 1$, and $s(j) = s(i_k)$ for every $j \in N_k$.*

The priority structure is ***acyclic*** *if there does not exist any cycle.*

By construction, a cycle is also an E-cycle. On the other hand, if $|A| \geq 3$ and $|N| \leq \sum_{a \in A} q_a$, we can easily embed any E-cycle with three agents into a stable allotment (N, s) that includes all agents to obtain a cycle, i.e., acyclicity is equivalent to E-acyclicity in this case. They are both satisfied by any priority structure if $|A| = 2$ and $|N| \leq \sum_{a \in A} q_a$. Therefore, the two notions only diverge when $|N| > \sum_{a \in A} q_a$. Below we give an example to illustrate a cycle with its associated stable allotment. The example also shows that, unlike the case of E-cycles, the shortest cycle may involve more than three agents.

Example 2. *Suppose that $N = \{1, 2, 3, 4, 5\}$, $A = \{a, b, c, d\}$, and $q_x = 1$ for all $x \in A$. Consider the following priority structure.*

[2] So the only difference from their notion is that we require $s(i) \neq \emptyset$ for all $i \in I$. Both notions are defined independently of preferences. Another related concept that is in the same vein but depends on preferences is *top fair set* in Rong, Tang, and Zhang (2020), where it is shown in the classical model that DA can be decomposed as a procedure of iteratively eliminating top fair sets if and only if it is an efficient mechanism or the priority structure is E-acyclic.

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
1	3	4	2
2	4	1	5
3	5	5	1
5	1	2	3
4	2	3	4

Let $I = \{1,2,3,4\}$, $s(3) = a$, $s(4) = b$, $s(1) = c$ and $s(2) = d$. Then (I,s) is a stable allotment, and within it there is a cycle $1 \succ_a 2 \succ_a 3 \succ_b 4 \succ_c 1$. It can also be shown that it is the unique cycle. There are many other E-cycles, such as $1 \succ_a 2 \succ_a 3 \succ_b 1$ and $3 \succ_b 4 \succ_b 1 \succ_c 3$, which cannot be embedded into stable allotments to produce cycles.

We introduce the following additional incentive properties for agents, which will be shown to be equivalent to group strategy-proofness for DA. Example 1 can be similarly used to see that DA may satisfy them in the presence of E-cycles.

- Haeringer and Klijn (2009) consider the preference revelation games under DA.[3] Suppose that the true preferences of the agents are R. If they simultaneously report their preferences as $\bar{R}$,[4] then the outcome is determined by DA as $f^{DA}(\bar{R})$. A strategy profile $\bar{R}$ is a Nash equilibrium of this game if $f_i^{DA}(\bar{R}) R_i f_i^{DA}(\bar{R}'_i, \bar{R}_{-i})$ for all $i \in N$ and $\bar{R}'_i$. Let $\mathscr{E}(R)$ be the set of all Nash equilibria, $\mathscr{O}(R) = \left\{ f^{DA}(\bar{R}) : \bar{R} \in \mathscr{E}(R) \right\}$ be the set of equilibrium outcomes, and $\mathscr{S}(R)$ be the set of stable allocations for R. f^{DA} implements the stable allocation correspondence in Nash equilibria if $\mathscr{O}(R) = \mathscr{S}(R)$ for all R.

- Kojima (2011) considers combined manipulations, where an agent misreports her preferences and then blocks the resulting allocation. A mechanism f is **robustly stable** if it is stable, strategy-proof, and there do not exist R, $i \in N$, R'_i and $a \in A$ such that $a P_i f_i(R)$, and (1) $\left| \left\{ j \in N : f_j(R'_i, R_{-i}) = a \right\} \right| < q_a$ or (2) $f_j(R'_i, R_{-i}) = a$ for some $j \in N$ with $i \succ_a j$.

- Afacan (2012) further considers combined manipulations by groups of agents. For each $a \in A$, let $\succeq_a^r$ be any complete, transitive and antisymmetric relation over 2^N that is *responsive* to $\succeq_a$, with $\succ_a^r$ denoting its asymmetric component.[5] Then for

[3] Their main focus is on the case where each agent may only be able to report a constrained preference list. Our results do not hold if an agent cannot report a full list.

[4] We also require $\emptyset$ to be the last option in the reported preferences.

[5] That is, for any $I \subseteq N$ and $i, j \in N \setminus I$, we have: (1) $I \cup \{i\} \succ_a^r I$; (2) $i \succ_a j$ implies $I \cup \{i\} \succ_a^r I \cup \{j\}$.

any $I \subseteq N$, define a's choice $C_a(I) \subseteq I$, such that $|C_a(I)| \leq q_a$ and $C_a(I) \succeq_a^r I'$ for any $I' \subseteq I$ with $|I'| \leq q_a$. Under a mechanism f, there is a *combined manipulation by a non-empty $I \subseteq N$ at R* if there exist R'_I, a partition of I as $I_1, \ldots, I_n$, and distinct $a_1, \ldots, a_n \in A$ such that for each $k \in \{1, \ldots, n\}$ and $i \in I_k$, we have $a_k P_i f_i(R)$ and

$$I_k \subseteq C_{a_k}\Big(\big\{j \in N : f_j(R'_I, R_{-I}) = a_k\big\} \cup I_k\Big).$$

Then a mechanism f is **weakly group robustly stable** if it is stable, group strategy-proof, and under f there does not exist any combined manipulation at any R. A weakly group robustly stable mechanism is also robustly stable.

We are ready to present the main result of the paper.

Theorem 1. *The following statements are equivalent:*

(i) f^{DA} *is group strategy-proof.*

(ii) f^{DA} *implements the stable allocation correspondence in Nash equilibria.*

(iii) f^{DA} *is robustly stable.*

(iv) f^{DA} *is weakly group robustly stable.*

(v) The priority structure $\succeq$ *is acyclic.*

In the classical model it has been shown by the above studies that DA satisfies any of these properties if and only if $\succeq$ is E-acyclic. Theorem 1 first extends these results. This is because if some $o \in A$ with $q_o = |N|$ assumes the role of outside option, then $|N| \leq \sum_{a\in A} q_a$ and thus acyclicity is equivalent to E-acyclicity.[6] In addition, note that in the special case of $|A| = 2$, DA always satisfies the incentive properties.

The main implication of the theorem is that, when $|N| > \sum_{a\in A} q_a$ (and hence there is no outside option), the necessary and sufficient condition for these equivalent properties becomes less restrictive, which also has notably different interpretations, compared to E-acyclicity. First, the following result of Ergin (2002) says $\succeq$ is E-acyclic if and only if any agent's ranks at two objects differ by at most one when she is not ranked high enough.

Lemma 2 (Ergin, 2002)**.** *The priority structure* $\succeq$ *is E-acyclic if and only if for any* $a, b \in A$ *and* $i \in N$ *with* $|U(\succ_a, i)| \geq q_a + q_b$*, we have* $\big||U(\succ_a, i)| - |U(\succ_b, i)|\big| \leq 1$.

Based on this, in the following proposition we characterize E-acyclic priorities in a slightly more explicit way. For brevity, we abuse the notation in this part and write $I \succ_a I'$ when $I, I' \subseteq N$ and $i \succ_a j$ for any $i \in I$ and any $j \in I'$.

[6] In this case the priority ordering of o is irrelevant. Moreover, the E-acyclicity of $(\succeq_a)_{a\in A}$ is equivalent to the E-acyclicity of $(\succeq_a)_{a\in A\setminus\{o\}}$, as the large capacity of o implies that it does not appear in any E-cycle.

Proposition 1. *The priority structure $\succeq$ is E-acyclic if and only if for any $a, b \in A$ there exists (possibly empty) $I \subseteq N$ satisfying the following conditions:*

- $|I| \leq q_a + q_b$.
- $I \succ_a N \setminus I$ *and* $I \succ_b N \setminus I$.
- *When $N \setminus I$ is non-empty, there is a partition of $N \setminus I$ as $I_1, \dots, I_n$ such that $|I_k| \leq 2$ for each $k \in \{1, \dots, n\}$, and $I_1 \succ_c I_2 \succ_c, \dots, \succ_c I_n$ for each $c \in \{a, b\}$.*

That is, for any $a, b \in A$, while there is no restriction on the priorities within a top group that has no more than $q_a + q_b$ agents, the ranks of all remaining agents must be sufficiently similar at these two objects. In contrast, if the total demand exceeds the total supply, acyclicity is satisfied at least under the circumstances where the ranks of such remaining agents are either sufficiently similar or sufficiently different.

Proposition 2. *Assume $|N| > \sum_{a \in A} q_a$. The priority structure $\succeq$ is acyclic if for any $a, b \in A$ there exists (possibly empty) $I \subseteq N$ satisfying the following conditions:*

- $|I| \leq q_a + q_b$.
- $I \succ_a N \setminus I$ *and* $I \succ_b N \setminus I$.
- *When $N \setminus I$ is non-empty, there is a partition of $N \setminus I$ as $I_1, \dots, I_n$ such that $|I_k| \leq 2$ for each $k \in \{1, \dots, n\}$, $I_1 \succ_a I_2 \succ_a, \dots, \succ_a I_n$, and*

$$\textit{either } I_1 \succ_b I_2 \succ_b, \dots, \succ_b I_n \ \textit{ or } \ I_n \succ_b I_{n-1} \succ_b, \dots, \succ_b I_1.$$

Intuitively, ranking the agents $N \setminus I$ similarly helps rule out E-cycles, while ranking them differently limits the formation of a stable allotment that incorporates agents in an E-cycle.

Finally, as a simple example, suppose $|N| > \sum_{a \in A} q_a$ and there is a given order of the agents. If every object ranks agents using this order, it is clear that under such homogenous priority structure DA is group strategy-proof, and satisfies all other desiderata considered in the paper. The above proposition indicates that, in addition, if some objects rank agents using this order, while other objects rank agents using the reversed order, then DA is still group strategy-proof, weakly group robustly stable, and implements the stable allocation correspondence in Nash equilibria.

4 Other Axioms

Acyclicity is not sufficient for DA to be efficient, which is illustrated by Example 1. As shown below, it turns out that we need a stronger restriction on priorities. Hence, efficiency is in general stronger than group strategy-proofness for the DA mechanism. This is in contrast to the fact that in the classical model any non-wasteful, individually rational and group strategy-proof mechanism (such as DA under E-acyclic priorities) is efficient.

On the other hand, for DA to be efficient, E-acyclicity is still sufficient but not necessary. For instance, we can add one additional agent to the problem in Example 1 so that she is ranked at the top by both objects. Then DA is efficient, as the E-cycle that consists of the three original agents does not induce any rejection cycle. We introduce the following new condition for efficiency. It rules out rejection cycles in the DA procedure that lead to welfare loss, and thus its role is the same as that of E-acyclicity in the classical model.

Definition 3. *A **weak cycle** consists of three distinct agents $i, j, k \in N$ such that the following conditions are satisfied:*

- *There exists a stable allotment (I, s) with $i, k \in I$, and $s(i) \neq s(k)$.*
- *(Cycle condition) $i \succ_{s(i)} j \succ_{s(i)} k \succ_{s(k)} i$.*
- *(Scarcity condition) There exist (possibly empty) disjoint sets $N_i, N_k \subseteq I \setminus \{i, j, k\}$ such that $N_i \subseteq U(\succ_{s(i)}, j)$, $|N_i| = q_{s(i)} - 1$, $s(\ell) = s(i)$ for all $\ell \in N_i$, $N_k \subseteq U(\succ_{s(k)}, i)$, $|N_k| = q_{s(k)} - 1$, and $s(\ell) = s(k)$ for all $\ell \in N_k$.*

*The priority structure is **strongly acyclic** if there does not exist any weak cycle.*

Both cycles and weak cycles are defined by embedding E-cycles into stable allotments, but in different ways. It can be shown that there is a weak cycle when there is a cycle, and any weak cycle is an E-cycle.[7] Therefore, E-acyclicity implies strong acyclicity, and the latter implies acyclicity. The three are all equivalent when $|N| \leq \sum_{a \in A} q_a$.

We extend our baseline model to further consider the consistency in DA outcomes or its robustness to non-simultaneous assignment. Given a problem R, an allocation μ

[7] In the proof of Theorem 2 in Appendix A.3 we introduce a *generalized weak cycle* that may involve more than three agents. Using Lemma 4 there, after adjusting the associated stable allotment a cycle becomes a generalized weak cycle. Then Lemma 5 shows that there is a weak cycle when there is a generalized weak cycle.

and a non-empty $N' \subseteq N$, if the agents $N \setminus N'$ leave the problem with their assignments under μ, then the reduced problem consists of $N', A', (q'_a)_{a\in A'}, \big(\succeq_a|_{N'}\big)_{a\in A'}$ and $\big(R_i|_{A'}\big)_{i\in N'}$, where $q'_a = q_a - |\mu^{-1}(a) \setminus N'|$ for each $a \in A$ and $A' = \{a \in A : q'_a > 0\}$. We simply denote this reduced problem as $R_{N',\mu}$. Suppose that $\bar{f}$ is an **extended mechanism** that chooses an allocation for every such reduced problem, then it is **consistent** if for any R and non-empty $N' \subseteq N$ we have $\bar{f}_i(R_{N',\bar{f}(R)}) = \bar{f}_i(R)$ for all $i \in N'$. Let $\bar{f}^{DA}$ denote the extended DA mechanism.

Theorem 2. *The following statements are equivalent:*

(i) f^{DA} *is efficient.*

(ii) $\bar{f}^{DA}$ *is consistent.*[8]

(iii) The priority structure $\succeq$ *is strongly acyclic.*

Finally, Kesten (2012) considers objects' incentives to truthfully report their capacities. Suppose that each $a \in A$ also has a preference relation R_a over subsets of agents. The preferences of agents and objects $R = (R_x)_{x\in N\cup A}$ as well as the capacities $q = (q_a)_{a\in A}$ are private information. In addition, there are exogenously given *minimum capacities* $\underline{q} = (\underline{q_a})_{a\in A}$. Then a mechanism f chooses an allocation for each R and q with $q \geq \underline{q}$. It is *non-manipulable via capacities* if for any R, $q \geq \underline{q}$, $a \in A$ and $\underline{q_a} \leq q'_a < q_a$, $f_a(R,q) R_a f_a(R, q'_a, q_{-a})$.

In the classical setup where $\emptyset$ is the outside option that can be preferred to some objects by each agent, Kesten (2012) shows that DA is non-manipulable via capacities if and only if the priority structure $\succeq$ with respect to $\underline{q}$ is E-acyclic. It is straightforward to show that this result holds in our case where $\emptyset$ is the last option for all agents, due to the flexible true capacities. In particular, when $|A| \geq 3$, to see the necessity of E-acyclicity, suppose there is an E-cycle that involves three agents and two objects. We can make a third object's true capacity large enough so that it serves the role of the outside option in the proof of Kesten (2012). Then by the same arguments some object in the E-cycle would manipulate via under-reporting its capacity.

[8] In the classical model, Ergin (2002) shows that for any given preference profile, if the DA outcome is efficient, then after some agents leave with their assignments, applying DA to the reduced problem gives each remaining agent the same assignment as before. This immediately implies "(i) $\Longrightarrow$ (ii)" in Theorem 2.

5 Conclusion

In the model of priority-based allocation where there may or may not be outside options, we have identified the necessary and sufficient conditions for DA to satisfy various properties. Both new conditions, acyclicity and strong acyclicity, are developed from E-acyclicity using stable allotments. Previous research on priority characterization has led to the conclusion that similar priority rankings across objects, and thus the absence of rejection cycles, are essential for several properties of DA regarding agent incentives. However, it is no longer the case when resources are very scarce, in the sense that agents do not have other options than the objects to be allocated, and there are more agents than the total capacities of the objects.

Appendix

A Proofs

A.1 Proof of Theorem 1

We show that (v) $\Longrightarrow$ (i), (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii), and (iii) $\Longrightarrow$ (v). Given that (iv) $\Longrightarrow$ (iii), we finish the proof by showing that (i) and (iii) together imply (iv).

We first state the following basic property of any strategy-proof mechanism, which will be useful in the proofs.

Lemma 3. *Suppose that f is strategy-proof. For any R, $i \in N$ and R'_i such that $\left\{a \in A : aR'_i f_i(R)\right\} \subseteq \left\{a \in A : aR_i f_i(R)\right\}$, we have $f_i(R) = f_i(R'_i, R_{-i})$.*

A.1.1 (v) $\Longrightarrow$ (i)

Suppose that f^{DA} is not group strategy-proof. Since it is strategy-proof, by Lemma 1, it is bossy. We first show the claim below.

Claim 1. *There exist R, R', $i \in N$, and $a, b \in A$ such that the following is satisfied:*

- *$R'_i \neq R_i$, and $R'_j = R_j$ for all $j \in N \setminus \{i\}$.*
- *$aP_i b$ and there does not exist c with $aP_i cP_i b$.*

- *R'_i is obtained from R_i by only pushing up b by one position (i.e., switching the positions of a and b in R_i), and thus $bP'_i a$.*

- *$f_i^{DA}(R) = f_i^{DA}(R') = b$, and $f^{DA}(R) \neq f^{DA}(R')$.*

Proof. Since f^{DA} is bossy, there exist R, i and R'_i such that $f_i^{DA}(R) = f_i^{DA}(R'_i, R_{-i})$ and $f^{DA}(R) \neq f^{DA}(R'_i, R_{-i})$. If $f_i^{DA}(R) = \emptyset$, then it is straightforward to see that $f^{DA}(R)$ is a stable allocation for (R'_i, R_{-i}) and $f^{DA}(R'_i, R_{-i})$ is a stable allocation for R, which implies the equivalence of the two allocations by the agent-optimal stability of DA. Therefore, let $f_i^{DA}(R) = b \in A$. Next, consider the process of pushing up the position of b by one position at a time in the preference list R_i, and the process of pushing up the position of b by one position at a time in R'_i. Suppose that $\bar{R}_i$ and $\bar{R}'_i$ are obtained after b is pushed up to the top in these two processes, respectively. During either process, agent i is always assigned b by Lemma 3. In particular, $f_i^{DA}(\bar{R}_i, R_{-i}) = f_i^{DA}(\bar{R}'_i, R_{-i}) = b$. The DA procedure for $(\bar{R}_i, R_{-i})$ is the same as the one for $(\bar{R}'_i, R_{-i})$ and thus $f^{DA}(\bar{R}_i, R_{-i}) = f^{DA}(\bar{R}'_i, R_{-i})$. Therefore, given that $f^{DA}(R) \neq f^{DA}(R'_i, R_{-i})$, there is some other agent whose assignment under DA changes at some point in at least one process. This finishes the proof of the claim. $\square$

Let R, R', $i \in N$, and $a, b \in A$ be specified as in Claim 1. For simplicity, denote $f^{DA}(R) = \mu$ and $f^{DA}(R') = \mu'$. It can be easily checked that μ is also stable for R', and thus μ' Pareto dominates μ for R'. By Theorem 1 of Erdil and Ergin (2008), there exists a *stable improvement cycle* for μ and R'. That is, there exists a list of $n \geq 2$ distinct agents $(i_1, \ldots, i_n)$ such that for each $k \in \{1, \ldots, n\}$, $\mu(i_k) \in A$, $\mu(i_{k+1}) P'_{i_k} \mu(i_k)$, and $i_k \succ_{\mu(i_{k+1})} j$ for all $j \in N$ with $\mu(i_{k+1}) P'_j \mu(j)$, where $i_{n+1} = i_1$. As priorities are strict and the agents in the cycle are distinct, the objects involved in the cycle, $\mu(i_1), \ldots, \mu(i_n)$, are also distinct. By the stability of μ for R', $i_{k+1} \succ_{\mu(i_{k+1})} i_k$ for all k. Moreover, let ν be the allocation obtained by carrying out the exchanges in the stable improvement cycle, i.e., $\nu(i_k) = \mu(i_{k+1})$ for all $k \in \{1, \ldots, n\}$, and $\nu(j) = \mu(j)$ for any other agent j. Then ν is stable for R'.

By the construction of R and R', we have

$$\left\{j \in N : cP'_j \mu(j)\right\} = \left\{j \in N : cP_j \mu(j)\right\}, \ \forall c \in A \setminus \{a\}, \tag{1}$$

and

$$\left\{j \in N : aP'_j \mu(j)\right\} \cup \{i\} = \left\{j \in N : aP_j \mu(j)\right\}. \tag{2}$$

If object a is not involved in the stable improvement cycle, then, by (1), $(i_1, \ldots, i_n)$ is also a stable improvement cycle for μ and R, and thus ν is a stable allocation for R that Pareto dominates μ, which leads to a contradiction. Hence, without loss of generality, let $\mu(i_1) = a$. Recall that $i_1 \succ_a i_n$. Since $\mu(i) = b$ and $bP'_i a$, $i_n \neq i$. Moreover, $aP_i b$ indicates $i_1 \succ_a i$ by the stability of μ for R. If $i_n \succ_a i$, then by (2) i_n is still ranked higher by a than any other agent j who prefers a to $\mu(j)$ under R. This fact in conjunction with (1) implies that $(i_1, \ldots, i_n)$ is a stable improvement cycle for μ and R, and a contradiction is reached. Therefore, we have $i_1 \succ_a i \succ_a i_n$.

The object b is not involved in the stable improvement cycle, and hence agent i is not in the cycle. To see this, suppose that $\mu(i_k) = b$ for some $k \in \{2, \ldots, n\}$. By (2) and the fact that $i \succ_a i_n$, i is ranked higher by a than any other agent j who prefers a to $\mu(j)$ under R. It follows that $(i_1, \ldots, i_{k-1}, i)$ is a stable improvement cycle for μ and R, which leads to a contradiction.

In sum, we have $n+1$ distinct agents $i_1, \ldots, i_n, i$ such that

$$i_1 \succ_a i \succ_a i_n \succ_{\mu(i_n)} i_{n-1} \ldots i_2 \succ_{\mu(i_2)} i_1.$$

To show that they satisfy the definition of a cycle, consider the stable allotment $(I, \nu|_I)$, where $I = \left\{j \in N : \nu(j) \neq \emptyset\right\}$, and $\{i_1, \ldots, i_n, i\} \subseteq I$. Recall that $\nu(i_k) = \mu(i_{k+1})$ for each $k \in \{1, \ldots, n\}$, $\nu(i) = b$, and all these objects are distinct. Finally, it is straightforward to see that the scarcity condition is satisfied by constructing $N_k = \left\{j \in N : \nu(j) = \nu(i_k), j \neq i_k\right\}$ for each $k \in \{1, \ldots, n\}$.

A.1.2 (i) $\Longrightarrow$ (ii)

Suppose that f^{DA} is group strategy-proof. Consider any preference profile R. First, to see that $\mathscr{S}(R) \subseteq \mathscr{O}(R)$, pick any $\mu \in \mathscr{S}(R)$. Construct a strategy $\bar{R}_i$ for each $i \in N$, such that $\mu(i)$ is the top choice in $\bar{R}_i$ if $\mu(i) \neq \emptyset$. Then, by the stability of μ for R, it is straightforward to see that $f^{DA}(\bar{R}) = \mu$ and $f_i^{DA}(\bar{R}) R_i f_i^{DA}(\bar{R}'_i, \bar{R}_{-i})$ for any i and $\bar{R}'_i$. Therefore, $\bar{R} \in \mathscr{E}(R)$ and $\mu \in \mathscr{O}(R)$.

On the other hand, suppose that $\mathscr{O}(R) \nsubseteq \mathscr{S}(R)$. That is, there exists a strategy profile $\bar{R} \in \mathscr{E}(R)$ such that $f^{DA}(\bar{R})$ is not stable for R. Denote $f^{DA}(\bar{R}) = \mu$. Then we can find $i \in N$ and $a \in A$ such that $aP_i\mu(i)$, and either $|\mu^{-1}(a)| < q_a$ or $\mu(j) = a$ for some $j \in N$ with $i \succ_a j$. Since μ is stable for $\bar{R}$, $\mu(i)\bar{P}_i a$. Let $\mu(i) = b \in A$, and $\bar{R}'_i$ be any preference relation with $a\bar{P}'_i b \bar{P}'_i c$ for all $c \in A \setminus \{a, b\}$. Then $\bar{R} \in \mathscr{E}(R)$ implies $f_i^{DA}(\bar{R}'_i, \bar{R}_{-i}) \neq a$. On the other hand, the strategy-proofness of f^{DA} implies $f_i^{DA}(\bar{R}'_i, \bar{R}_{-i}) \bar{R}'_i b$. Therefore, $f_i^{DA}(\bar{R}'_i, \bar{R}_{-i}) = b$.

Then by the non-bossiness of f^{DA}, $f^{DA}(\bar{R}'_i, \bar{R}_{-i}) = \mu$. However, as a is the top choice in $\bar{R}'_i$, this shows that $f^{DA}(\bar{R}'_i, \bar{R}_{-i})$ is not stable for $(\bar{R}'_i, \bar{R}_{-i})$, and thus a contradiction is reached.

A.1.3 (ii) $\Longrightarrow$ (iii)

Suppose that $\mathcal{O}(R) = \mathcal{S}(R)$ for all R, but f^{DA} is not robustly stable. Since it is strategy-proof and stable, there exist R, $i \in N$, R'_i and $a \in A$ such that $aP_i f_i^{DA}(R)$, and $\left|\left\{j \in N : f_j^{DA}(R'_i, R_{-i}) = a\right\}\right| < q_a$ or $f_j^{DA}(R'_i, R_{-i}) = a$ for some $j \in N$ with $i \succ_a j$. Then by the stability of $f^{DA}(R'_i, R_{-i})$ for (R'_i, R_{-i}), $f_i^{DA}(R'_i, R_{-i}) = b$ for some $b \in A$. By the strategy-proofness of f^{DA}, $b \neq a$. Construct a preference relation R''_i such that $aP''_i bP''_i c$ for all $c \in A \setminus \{a, b\}$. Given that f^{DA} is strategy-proof, $aP_i f_i^{DA}(R)$ implies $f_i^{DA}(R''_i, R_{-i}) \neq a$, and $f_i^{DA}(R'_i, R_{-i}) = b$ implies $f_i^{DA}(R''_i, R_{-i}) R''_i b$. Therefore, $f_i^{DA}(R''_i, R_{-i}) = b$.

Consider the preference revelation game where the true preferences are (R''_i, R_{-i}). By strategy-proofness, no agent can be strictly better-off by misreporting. Therefore, $(R''_i, R_{-i}) \in \mathcal{E}(R''_i, R_{-i})$. Since $f_i^{DA}(R'_i, R_{-i}) = f_i^{DA}(R''_i, R_{-i})$, we also have $(R'_i, R_{-i}) \in \mathcal{E}(R''_i, R_{-i})$. However, as a is the top choice in R''_i, $f^{DA}(R'_i, R_{-i})$ is not stable for (R''_i, R_{-i}), and thus a contradiction is reached.

A.1.4 (iii) $\Longrightarrow$ (v)

Suppose that there exists a cycle that consists of distinct $i_1, \ldots, i_n, i \in N$ such that $n \geq 2$ and the following conditions are satisfied:

- There is a stable allotment (I, s) with $i_1, \ldots, i_n, i \in I$, and $s(i_1), \ldots, s(i_n), s(i)$ are distinct.
- $i_1 \succ_{s(i_2)} i \succ_{s(i_2)} i_2$, and $i_k \succ_{s(i_{k+1})} i_{k+1}$ for all $k \in \{2, \ldots, n\}$, where $i_{n+1} = i_1$.
- There are mutually disjoint $N_1, \ldots, N_n \subseteq I \setminus \{i_1, \ldots, i_n, i\}$ such that $N_2 \subseteq U(\succ_{s(i_2)}, i)$, and $N_k \subseteq U(\succ_{s(i_k)}, i_k)$ for $k \neq 2$. Moreover, for all $k \in \{1, \ldots, n\}$, $|N_k| = q_{s(i_k)} - 1$, and $s(j) = s(i_k)$ for every $j \in N_k$.

Consider a preference profile R such that

- $R_i : s(i_2), \ldots,$
- $R_{i_k} : s(i_k), s(i_{k+1}), \ldots,\ \forall k \in \{1, \ldots, n\}$,
- $R_j : s(j), \ldots,\ \forall j \in I \setminus \{i_1, \ldots, i_n, i\}$.

First, suppose $f_i^{DA}(R) = s(i_2)$. Since $N_2 \subseteq U(\succ_{s(i_2)}, i)$, $|N_2| = q_{s(i_2)} - 1$ and $i_1 \succ_{s(i_2)} i$, by stability, we have $f_j^{DA}(R) = s(i_2)$ for all $j \in N_2$ and $f_{i_1}^{DA}(R) = s(i_1)$. It can be similarly shown that stability further implies $f_{i_n}^{DA}(R) = s(i_n), \ldots, f_{i_2}^{DA}(R) = s(i_2)$. Then there are at least $q_{s(i_2)} + 1$ agents who receive $s(i_2)$, which is impossible.

Therefore, $f_i^{DA}(R) \neq s(i_2)$. Let R'_i be any preference relation in which $s(i)$ is ranked at the top. Then under the profile (R'_i, R_{-i}), $s(j)$ is j's first choice for every $j \in I$, and thus, by the definition of a stable allotment, $f_j^{DA}(R'_i, R_{-i}) = s(j)$ for every $j \in I$. In particular, $f_{i_2}^{DA}(R'_i, R_{-i}) = s(i_2)$. Given that $s(i_2) P_i f_i^{DA}(R)$ and $i \succ_{s(i_2)} i_2$, f^{DA} is not robustly stable.

A.1.5 (i) and (iii) $\Longrightarrow$ (iv)

Suppose that f^{DA} is group strategy-proof and robustly stable, but it is not weakly group robustly stable. Then there exists a combined manipulation by some group of agents. Let $I \subseteq N$ be one of the smallest possible manipulating groups, who can do a combined manipulation at R. That is, if there is a combined manipulation by some $I' \subseteq N$ at some preference profile, then $|I| \leq |I'|$. Moreover, there exist R'_I, a partition of I as $I_1, \ldots, I_n$, and distinct $a_1, \ldots, a_n$ such that for each k and $i \in I_k$, we have $a_k P_i f_i^{DA}(R)$ and $I_k \subseteq C_{a_k}\Big(\big\{j \in N : f_j^{DA}(R'_I, R_{-I}) = a_k\big\} \cup I_k\Big)$. By robust stability, $|I| \geq 2$.

We first prove the following result.

Claim 2. *There do not exist an agent $i \in I$ and R''_i such that $f_i^{DA}(R''_i, R_{-i}) = f_i^{DA}(R)$ and $f_i^{DA}(R''_i, R'_{I\setminus\{i\}}, R_{-I}) = f_i^{DA}(R'_I, R_{-I})$.*

Proof. Suppose that there exist such agent $i \in I$ and R''_i. Since f^{DA} is non-bossy, we have $f^{DA}(R''_i, R_{-i}) = f^{DA}(R)$ and $f^{DA}(R''_i, R'_{I\setminus\{i\}}, R_{-I}) = f^{DA}(R'_I, R_{-I})$. Note that for any $a \in A$ and $I' \subseteq N$, by the definition of $C_a(I')$, the object a chooses I' if $|I'| \leq q_a$, and it chooses the highest ranked q_a agents from I' if $|I'| > q_a$. The above facts then indicate that there is a combined manipulation by $I \setminus \{i\}$ at the preference profile (R''_i, R_{-i}), where $I \setminus \{i\}$ can misreport their preferences as $R'_{I\setminus\{i\}}$ and then jointly block the resulting allocation. This contradicts to our initial choice of I. $\square$

If $f_i^{DA}(R) = \emptyset$ for some $i \in I$, then by Lemma 3, $f_i^{DA}(R'_i, R_{-i}) = \emptyset$. By setting $R''_i = R'_i$, we have $f_i^{DA}(R''_i, R_{-i}) = f_i^{DA}(R)$ and $f_i^{DA}(R''_i, R'_{I\setminus\{i\}}, R_{-I}) = f_i^{DA}(R'_I, R_{-I})$, contradicting to the above claim. Therefore, $f_i^{DA}(R) \neq \emptyset$ for all $i \in I$. Similarly, it can be seen that $f_i^{DA}(R'_I, R_{-I}) \neq \emptyset$ for all $i \in I$.

Consider the deferred acceptance procedure for R. Pick $i \in I$ such that i is not assigned earlier than any other agent in I. That is, if i applies to $f_i^{DA}(R)$ in step ℓ of DA, then any $j \in I$ applies to $f_j^{DA}(R)$ in step $\ell' \leq \ell$. Let $i \in I_k$, $a_k = a$, $f_i^{DA}(R) = b \in A$, and $f_i^{DA}(R'_I, R_{-I}) = c \in A$. Then it is already known that $aP_i b$. Suppose that $cR_i b$. Push the position of c to the top in the preference relation R_i, and denote the resulting relation as R_i^1. Then by Lemma 3, $f_i^{DA}(R_i^1, R_{-i}) = f_i^{DA}(R) = b$ and $f_i^{DA}(R_i^1, R'_{I\setminus\{i\}}, R_{-I}) = f_i^{DA}(R'_I, R_{-I}) = c$, contradicting to Claim 2. Therefore, we have $aP_i bP_i c$.

Given R_i, move up the position of c such that it is just above b, and denote the resulting relation as R_i^2.[9] By strategy-proofness, $f_i^{DA}(R_i^2, R_{-i}) \in \{b, c\}$. In addition, construct R_i^3 by pushing the position of c to the top in R_i^2.

If $f_i^{DA}(R_i^2, R_{-i}) = b$, then by Lemma 3, we have $f_i^{DA}(R_i^3, R_{-i}) = f_i^{DA}(R) = b$, and $f_i^{DA}(R_i^3, R'_{I\setminus\{i\}}, R_{-I}) = f_i^{DA}(R'_I, R_{-I}) = c$, contradicting to Claim 2.

Finally, consider the case where $f_i^{DA}(R_i^2, R_{-i}) = c$. By our initial choice of i, every $j \in I \setminus \{i\}$ will still be rejected by any object better than $f_j^{DA}(R)$ in the deferred acceptance procedure for (R_i^2, R_{-i}). It follows that $f_j^{DA}(R) R_j f_j^{DA}(R_i^2, R_{-i})$ for all $j \in I \setminus \{i\}$. Next, consider R_i^3. By Lemma 3,

$$f_i^{DA}(R_i^3, R_{-i}) = f_i^{DA}(R_i^2, R_{-i}) = f_i^{DA}(R_i^3, R'_{I\setminus\{i\}}, R_{-I}) = c.$$

Since f^{DA} is non-bossy, we have $f^{DA}(R_i^3, R_{-i}) = f^{DA}(R_i^2, R_{-i})$, and $f^{DA}(R_i^3, R'_{I\setminus\{i\}}, R_{-I}) = f^{DA}(R'_I, R_{-I})$. The former indicates $f_j^{DA}(R) R_j f_j^{DA}(R_i^3, R_{-i})$ for all $j \in I \setminus \{i\}$, which further implies that if $j \in I_{k'} \setminus \{i\}$, then $a_{k'} P_j f_j^{DA}(R_i^3, R_{-i})$. Therefore, similar to the proof of Claim 2, there is a combined manipulation by $I \setminus \{i\}$ at (R_i^3, R_{-i}), where the agents $I \setminus \{i\}$ can misreport their preferences as $R'_{I\setminus\{i\}}$ and jointly block the resulting allocation. This contradicts to our choice of I and finishes the proof.

A.2 Proofs of Propositions 1 and 2

A.2.1 Proof of Proposition 1

The "if" part follows immediately from Lemma 2. To show the "only if" part, assume that the priority structure $\succeq$ is E-acyclic. We first prove the following claim.

Claim 3. *Consider any $a, b \in A$ and $i \in N$ such that $|U(\succ_a, i)| \geq q_a + q_b$ and $|U(\succ_b, i)| = |U(\succ_a, i)| + 1$. If $|U(\succ_b, j)| = |U(\succ_a, i)|$, then $|U(\succ_a, j)| = |U(\succ_b, i)|$.*

Proof. Suppose that $|U(\succ_a, i)| \geq q_a + q_b$ and $|U(\succ_b, i)| = |U(\succ_a, i)| + 1$. Then there exists $j \in N$ such that $j \succ_b i$ and $i \succ_a j$. Since $|U(\succ_a, j)| \geq q_a + q_b$, Lemma 2 indicates that the

[9] So there is no $d \in A$ such that $cP_i^2 dP_i^2 b$.

ranks of j at a and b differ by at most one. This implies that j must be ranked one position lower than i by a, and one position higher than i by b. That is, $|U(\succ_b, j)| = |U(\succ_a, i)|$ and $|U(\succ_a, j)| = |U(\succ_b, i)|$. $\square$

Consider any $a, b \in A$. When $|N| \leq q_a + q_b$, it is clear that we can choose $I = N$ which satisfies the three conditions in the proposition. Suppose that $|N| > q_a + q_b$. Consider the agent i with $|U(\succ_a, i)| = q_a + q_b$. We will show that a set I satisfying the three conditions can be constructed in the following three cases.

Case 1: $|U(\succ_b, i)| = q_a + q_b$. Then Lemma 2 implies $U(\succ_a, i) = U(\succ_b, i)$. Let $I = U(\succ_a, i)$. Then we have $|I| \leq q_a + q_b$, $I \succ_a N \setminus I$ and $I \succ_b N \setminus I$. Moreover, using Claim 3, it is straightforward to partition $N \setminus I$ as $I_1, \ldots, I_n$ such that $I_1 = \{i\}$, $|I_k| \leq 2$ for each $k > 1$, and $I_1 \succ_c I_2 \succ_c, \ldots, \succ_c I_n$ for each $c \in \{a, b\}$.

Case 2: $|U(\succ_b, i)| = |U(\succ_a, i)| - 1$. Consider $j \in N$ with $|U(\succ_b, j)| = |U(\succ_a, i)|$. By applying Claim 3 to j, we cannot have $|U(\succ_a, j)| = |U(\succ_b, j)| + 1$. Therefore, by Lemma 2, $|U(\succ_a, j)| = |U(\succ_b, j)| - 1$. It also follows from Lemma 2 that $U(\succ_a, j) = U(\succ_b, i)$. Let $I = U(\succ_a, j)$. Then $N \setminus I$ can be partitioned using Claim 3 as in the previous case.

Case 3: $|U(\succ_b, i)| = |U(\succ_a, i)| + 1$. Consider $j \in N$ with $|U(\succ_b, j)| = |U(\succ_a, i)|$. By Claim 3, $|U(\succ_a, j)| = |U(\succ_b, i)|$. We choose $I = U(\succ_a, i) = U(\succ_b, j)$, which can be shown to satisfy the three conditions in the proposition.

A.2.2 Proof of Proposition 2

Assume that $|N| > \sum_{a \in A} q_a$, and $\succeq$ has the structure specified in the proposition. To prove by contradiction, suppose that $\succeq$ is not acyclic and there is a cycle that consists of distinct $i_1, \ldots, i_n, i \in N$ such that $n \geq 2$ and the following conditions are satisfied:

- There is a stable allotment (I, s) with $i_1, \ldots, i_n, i \in I$, and $s(i_1), \ldots, s(i_n), s(i)$ are distinct.
- $i_1 \succ_{s(i_2)} i \succ_{s(i_2)} i_2$, and $i_k \succ_{s(i_{k+1})} i_{k+1}$ for all $k \in \{2, \ldots, n\}$, where $i_{n+1} = i_1$.
- There are mutually disjoint $N_1, \ldots, N_n \subseteq I \setminus \{i_1, \ldots, i_n, i\}$ such that $N_2 \subseteq U(\succ_{s(i_2)}, i)$, and $N_k \subseteq U(\succ_{s(i_k)}, i_k)$ for $k \neq 2$. Moreover, for all $k \in \{1, \ldots, n\}$, $|N_k| = q_{s(i_k)} - 1$, and $s(j) = s(i_k)$ for every $j \in N_k$.

Without loss of generality, assume that it is one of the shortest cycles. If $i_n \succ_{s(i_2)} i$, then $n \geq 3$ and there is a shorter cycle in which $i_n \succ_{s(i_2)} i \succ_{s(i_2)} i_2 \succ_{s(i_3)} i_3 \ldots i_{n-1} \succ_{s(i_n)} i_n$, and the scarcity condition is satisfied by $N_2, \ldots, N_n$. Therefore, we have $i \succ_{s(i_2)} i_n$.

Denote $s(i_2) = a$ and $s(i_1) = b$. Then there exists $J \subseteq N$ such that:

- $|J| \leq q_a + q_b$.
- $J \succ_a N \setminus J$ and $J \succ_b N \setminus J$.
- When $N \setminus J \neq \emptyset$, there is a partition of $N \setminus J$ as $I_1, \ldots, I_m$ such that $|I_k| \leq 2$ for each $k \in \{1, \ldots, m\}$, $I_1 \succ_a I_2 \succ_a, \ldots, \succ_a I_m$, and

$$\text{either } I_1 \succ_b I_2 \succ_b, \ldots, \succ_b I_m \text{ or } I_m \succ_b I_{m-1} \succ_b, \ldots, \succ_b I_1.$$

If $i \in J$, then $\{i_1\} \cup N_2 \succ_a \{i\}$ implies $\{i_1\} \cup N_2 \subseteq J$. In addition, $\{i_n\} \cup N_1 \succ_b \{i_1\}$ implies $\{i_n\} \cup N_1 \subseteq J$. Then $|J| \geq |N_1| + |N_2| + |\{i_1, i, i_n\}| = q_a + q_b + 1$ and a contradiction is reached. Hence we have $i \notin J$. Then, it follows from $i \succ_a i_n$, $i \succ_a i_2$ and $i_n \succ_b i_1$ that $\{i, i_n, i_2, i_1\} \cap J = \emptyset$.

Since $i_1 \succ_a i \succ_a i_n$, there is no $k \in \{1, \ldots, m\}$ such that $I_k = \{i_1, i_n\}$. Then $i_n \succ_b i_1$ implies

$$I_1 \succ_a I_2 \succ_a, \ldots, \succ_a I_m \text{ and } I_m \succ_b I_{m-1} \succ_b, \ldots, \succ_b I_1.$$

Therefore, there does not exist an agent $j \in N$ such that $i_1 \succ_a i \succ_a i_2 \succ_a j$ and $i_1 \succ_b j$. We finish the proof of the proposition by showing that such agent j exists.

Consider any preference profile R where for every $j \in I$, the top choice in R_j is $s(j)$. Then $f_j^{DA}(R) = s(j)$ for every $j \in I$. In particular, $f_{i_2}^{DA}(R) = a$ and $f_{i_1}^{DA}(R) = b$. Since $|N| > \sum_{c \in A} q_c$, there exists $j \in N$ such that $f_j^{DA}(R) = \emptyset$. By the stability of $f^{DA}(R)$, we have $i_2 \succ_a j$ and $i_1 \succ_b j$.

A.3 Proof of Theorem 2

As mentioned in Section 4 (Footnote 8), it follows from Ergin (2002) that (i) $\Longrightarrow$ (ii). We will show that (iii) $\Longrightarrow$ (i) and (ii) $\Longrightarrow$ (iii).

A.3.1 (iii) $\Longrightarrow$ (i)

We introduce a more general concept than weak cycles. A **generalized weak cycle** consists of distinct agents $i_1, \ldots, i_n, i \in N$, where $n \geq 2$, such that the following conditions are satisfied:

- There exists a stable allotment (I,s) with $i_1,\ldots,i_n \in I$, and the objects $s(i_1),\ldots,s(i_n)$ are distinct.
- (Cycle condition) $i_1 \succ_{s(i_1)} i \succ_{s(i_1)} i_2$, and $i_k \succ_{s(i_k)} i_{k+1}$ for all $k \in \{2,\ldots,n\}$, where $i_{n+1}=i_1$.
- (Scarcity condition) There are (possibly empty) mutually disjoint sets $N_1,\ldots,N_n \subseteq I \setminus \{i_1,\ldots,i_n,i\}$ such that $N_1 \subseteq U(\succ_{s(i_1)},i)$, and $N_k \subseteq U(\succ_{s(i_k)},i_{k+1})$ for $k \neq 1$. Moreover, for all $k \in \{1,\ldots,n\}$, $|N_k| = q_{s(i_k)}-1$, and $s(j)=s(i_k)$ for every $j \in N_k$.

Suppose that there exists R such that $f^{DA}(R)$ is not efficient. We first extend the proof of the main theorem in Ergin (2002) to show that there exists a generalized weak cycle.

Let $f^{DA}(R)=\mu$. As shown by Ergin (2002), the inefficiency of μ implies the existence of an *exchange cycle* that consists of $n \geq 2$ distinct agents $i_1,\ldots,i_n$ such that the objects $\mu(i_1),\ldots,\mu(i_n)$ are distinct and for each $k \in \{1,\ldots,n\}$, $\mu(i_{k+1})P_{i_k}\mu(i_k)$, where $i_{n+1}=i_1$. Note that by stability

$$i_1 \succ_{\mu(i_1)} i_n \succ_{\mu(i_n)} i_{n-1} \ldots i_2 \succ_{\mu(i_2)} i_1.$$

For our purpose, without loss of generality, we make two assumptions on this exchange cycle. First, it is one of the shortest exchange cycles. Second, for each k, i_k cannot be replaced by an agent with a higher priority at $\mu(i_{k+1})$, i.e., there does not exist $i \in N \setminus \{i_1,\ldots,i_n\}$ such that $\mu(i)=\mu(i_k)$, $\mu(i_{k+1})P_i\mu(i)$ and $i \succ_{\mu(i_{k+1})} i_k$.[10]

Then, given the exchange cycle, Ergin (2002) shows that there exist $j \in N\setminus\{i_1,\ldots,i_n\}$ and $k \in \{1,\ldots,n\}$ such that $\mu(i_{k+1})P_j\mu(j)$ and $i_{k+1} \succ_{\mu(i_{k+1})} j \succ_{\mu(i_{k+1})} i_k$. Without loss of generality, let $k=n$. As $\mu(i_1)P_j\mu(j)$, $\mu(j) \neq \mu(i_1)$. Moreover, our first assumption on the exchange cycle implies $\mu(j) \neq \mu(i_\ell)$ for any $1 < \ell < n$, and the second one implies $\mu(j) \neq \mu(i_n)$. In sum, $\mu(j) \neq \mu(i_\ell)$ for any $\ell \in \{1,\ldots,n\}$. Therefore, after defining $I=\{i \in N : \mu(i) \neq \emptyset\}$, and $N_\ell = \{i \in N : \mu(i)=\mu(i_\ell), i \neq i_\ell\}$ for each ℓ, we have $N_1,\ldots,N_n \subseteq I \setminus \{i_1,\ldots,i_n,j\}$. Using the stable allotment $(I,\mu|_I)$, we have found a generalized weak cycle that consists of the agents $i_1,\ldots,i_n,j$, where

$$i_1 \succ_{\mu(i_1)} j \succ_{\mu(i_1)} i_n \succ_{\mu(i_n)} i_{n-1} \ldots i_2 \succ_{\mu(i_2)} i_1$$

and the scarcity condition is satisfied by $N_1,\ldots,N_n$.

[10] If there is such agent i, replacing i_k with i gives a different exchange cycle. We can repeat this operation until there is no further replacement so that the assumption is satisfied.

The rest of the proof consists of two lemmata. The first one presents a property of a stable allotment (Lemma 4), which can be easily shown using its definition. Based on this result, we show that, in general, there is a weak cycle whenever there is a generalized weak cycle (Lemma 5).

Lemma 4. *Given a stable allotment (I,s), and $n \geq 2$ distinct agents $i_1,\dots,i_n \in I$, define $s' : I \to A$ such that $s'(i_k) = s(i_{k+1})$ for all $k \in \{1,\dots,n\}$, where $i_{n+1} = i_1$, and $s'(j) = s(j)$ for all $j \in I \setminus \{i_1,\dots,i_n\}$. If there does not exist $j \in N \setminus I$ such that $j \succ_{s(i_{k+1})} i_k$ for any k, then (I,s') is a stable allotment.*

That is, a different stable allotment can be constructed by letting some agents exchange their original allotments, as long as each of them is ranked higher by the new allotment than any agent outside the stable allotment.

Lemma 5. *If there is a generalized weak cycle, then there is a weak cycle.*

Proof. We prove by contradiction. Suppose that one of the shortest generalized weak cycles consists of $n+1$ distinct agents $i_1,\dots,i_n,i$, where $n \geq 3$, such that $i_1,\dots,i_n$ are in a stable allotment (I,s), $s(i_1),\dots,s(i_n)$ are distinct, $i_1 \succ_{s(i_1)} i \succ_{s(i_1)} i_2$, $i_k \succ_{s(i_k)} i_{k+1}$ for all $k \in \{2,\dots,n\}$, where $i_{n+1} = i_1$, and the scarcity condition is satisfied by $N_1,\dots,N_n$.

Note that, for any $k \in \{1,\dots,n\}$, $\left|\left\{j \in I : s(j) = s(i_k)\right\}\right| = q_{s(i_k)}$, which implies that for any $j \in N_k \cup \{i_k\}$ and $\ell \in N \setminus I$, we have $j \succ_{s(i_k)} \ell$.

We finish the proof in the following five steps.

Step 1: $i_n \succ_{s(i_1)} j$ for all $j \in \{N \setminus I\} \cup \{i\}$.

To see this, suppose that $j \succ_{s(i_1)} i_n$ for some $j \in \{N \setminus I\} \cup \{i\}$. Then given the stable allotment (I,s), there exists a weak cycle in which $i_1 \succ_{s(i_1)} j \succ_{s(i_1)} i_n \succ_{s(i_n)} i_1$, and the scarcity condition is satisfied by N_1 and N_n, which leads to a contradiction.

Step 2: we can find $j^* \in N \setminus I$ such that $j^* \succeq_{s(i_n)} \ell$ for all $\ell \in N \setminus I$ and $j^* \succ_{s(i_n)} i_1$.

Suppose that there is no such agent j^*. Then $i_1 \succ_{s(i_n)} j$ for all $j \in N \setminus I$. Define s' such that $s'(i_1) = s(i_n)$, $s'(i_n) = s(i_1)$, and $s'(j) = s(j)$ for all $j \in I \setminus \{i_1, i_n\}$. Then by Step 1 and Lemma 4, (I,s') is a stable allotment. Given (I,s'), there is a generalized weak cycle in which $i_n \succ_{s'(i_n)} i \succ_{s'(i_n)} i_2 \succ_{s(i_2)} i_3 \dots i_{n-1} \succ_{s(i_{n-1})} i_n$ and the scarcity condition is satisfied by $N_1,\dots,N_{n-1}$. A contradiction is reached since this is a shorter generalized weak cycle.

Step 3: let $\ell^* \in N_{n-1} \cup \{i_{n-1}\}$ such that $j \succeq_{s(i_{n-1})} \ell^*$ for all $j \in N_{n-1} \cup \{i_{n-1}\}$. Then $i_1 \succ_{s(i_{n-1})} \ell^*$.

If $\ell^* \succ_{s(i_{n-1})} i_1$, then given the stable allotment (I,s), there is a generalized weak cycle with only n agents, in which $i_1 \succ_{s(i_1)} i \succ_{s(i_1)} i_2 \dots i_{n-1} \succ_{s(i_{n-1})} i_1$ and the scarcity condition is satisfied by $N_1, \dots, N_{n-1}$.

Step 4: $\ell^* \succ_{s(i_n)} j^*$.

If this is not true, then given the stable allotment (I,s), there is a weak cycle in which $i_n \succ_{s(i_n)} j^* \succ_{s(i_n)} \ell^* \succ_{s(i_{n-1})} i_n$ (note that $s(\ell^*) = s(i_{n-1})$), and the scarcity condition is satisfied by N_n and $\{N_{n-1} \cup \{i_{n-1}\}\} \setminus \{\ell^*\}$.

Step 5: there is a weak cycle and hence a contradiction is reached.

Define s' such that $s'(i_1) = s(i_{n-1})$, $s'(\ell^*) = s(i_n)$, $s'(i_n) = s(i_1)$, and $s'(j) = s(j)$ for all $j \in I \setminus \{i_1, \ell^*, i_n\}$. By Step 3, $i_1 \succ_{s(i_{n-1})} j$ for all $j \in N \setminus I$. By Step 4 as well as the choice of j^* in Step 2, $\ell^* \succ_{s(i_n)} j$ for all $j \in N \setminus I$. Then given Step 1, by Lemma 4, (I,s') is a stable allotment. Given (I,s'), we find a weak cycle in which $\ell^* \succ_{s(i_n)} j^* \succ_{s(i_n)} i_1 \succ_{s(i_{n-1})} \ell^*$, and the scarcity condition is satisfied by N_n and $\{N_{n-1} \cup \{i_{n-1}\}\} \setminus \{\ell^*\}$. □

A.3.2 (ii) $\Longrightarrow$ (iii)

Suppose that there exists a weak cycle that consists of distinct $i, j, k \in N$ such that the following conditions are satisfied:

- There is a stable allotment (I,s) with $i, k \in I$, and $s(i) \neq s(k)$.
- $i \succ_{s(i)} j \succ_{s(i)} k \succ_{s(k)} i$.
- There are disjoint $N_i, N_k \subseteq I \setminus \{i, j, k\}$ such that $N_i \subseteq U(\succ_{s(i)}, j)$, $|N_i| = q_{s(i)} - 1$, $s(\ell) = s(i)$ for all $\ell \in N_i$, $N_k \subseteq U(\succ_{s(k)}, i)$, $|N_k| = q_{s(k)} - 1$, and $s(\ell) = s(k)$ for all $\ell \in N_k$.

Note that if $j \in I$, then $s(j) \notin \{s(i), s(k)\}$.

Consider a preference profile R such that

- $R_i : s(k), s(i), \dots,$
- $R_j : s(i), s(j), \dots,$ if $j \in I$,
- $R_j : s(i), \dots,$ if $j \notin I$,
- $R_k : s(i), s(k), \dots,$

- $R_\ell : s(\ell), \dots, \forall \ell \in I \setminus \{i, j, k\}$.

Since $N_i \cup \{j\} \subseteq U(\succ_{s(i)}, k)$ and the first choice of every agent in $N_i \cup \{j\}$ is $s(i)$, by stability $f_k^{DA}(R) \neq s(i)$. Similarly, given that $N_k \cup \{k\} \subseteq U(\succ_{s(k)}, i)$ and $s(k) R_k f_k^{DA}(R)$, by stability $f_i^{DA}(R) \neq s(k)$. Therefore, neither i nor k can receive her first choice.

Let R' be a preference profile such that:

- The first choice in R'_i is $s(i)$, and the first choice in R'_k is $s(k)$.
- The first choice in R'_j is $s(j)$ if $j \in I$, and $R'_j = R_j$ otherwise.
- $R'_\ell = R_\ell$ for every $\ell \in N \setminus \{i, j, k\}$.

Under R', the first choice of ℓ is $s(\ell)$ for all $\ell \in I$. Therefore, $f_\ell^{DA}(R') = s(\ell)$ for all $\ell \in I$. Then, $f^{DA}(R')$ is also a stable allocation for R. To see this, suppose that for some $\ell \in N$ and $a \in A$ we have $a P_\ell f_\ell^{DA}(R')$, and, under $f^{DA}(R')$, a is not fully assigned or is assigned to some agent ℓ' with $\ell \succ_a \ell'$. Then by the stability of $f^{DA}(R')$ for R', we have $f_\ell^{DA}(R') P'_\ell a$. By the construction of R and R', this implies that $\ell = i$ and $a = s(k)$, $\ell = j \in I$ and $a = s(i)$, or $\ell = k$ and $a = s(i)$. However, under $f^{DA}(R')$, all the copies of $s(k)$ are assigned to $N_k \cup \{k\} \subseteq U(\succ_{s(k)}, i)$, and all the copies of $s(i)$ are assigned to $N_i \cup \{i\} \subseteq U(\succ_{s(i)}, j) \subseteq U(\succ_{s(i)}, k)$, which leads to a contradiction.

Since $f^{DA}(R)$ Pareto dominates $f^{DA}(R')$ for R, we have $f_i^{DA}(R) = s(i)$ and $f_k^{DA}(R) = s(k)$. Consider the extended DA, $\bar{f}^{DA}$, and the reduced problem $R_{\{i,k\}, \bar{f}^{DA}(R)}$. It is clear that $\bar{f}_i^{DA}(R_{\{i,k\}, \bar{f}^{DA}(R)}) = s(k) \neq \bar{f}_i^{DA}(R)$, and thus $\bar{f}^{DA}$ is not consistent.

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