Random Assignment Respecting Group Priorities*

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Abstract

When allocating indivisible objects, agents may be partitioned into several groups and different groups are granted different priority rights. Group-wise random serial dictatorship (GRSD) and group-wise probabilistic serial mechanism (GPS) can be applied to such problems to restore fairness within and across groups. GPS outperforms GRSD not only in efficiency but also in stability ex-ante. The family of group-wise eating mechanisms characterizes the set of sd-efficient and ex-ante stable assignments, while the family of asymmetric GRSDs characterizes the set of ex-post stable and efficient assignments. However, GRSD satisfies a mild consistency concept based on Bayesian update while GPS does not, and consequently the advantages of GPS may disappear in cases of non-simultaneous assignment. We finish by characterizing GPS and considering comparative statics results as the group structure becomes more dense.

1 Introduction

In a *house allocation* problem (Hylland and Zeckhauser, 1979), a set of indivisible objects (houses) has to be assigned to a set of agents without monetary transfer and each agent can be assigned at most one object.[1] Due to fairness considerations random allocation mechanisms are often used in practice. One of the most popular such mechanisms is the *random serial dictatorship* (RSD): an ordering of the agents is picked from the uniform

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[1] In a house allocation problem the agents collectively own all the houses. A related problem is the *housing market* (Shapley and Scarf, 1974), in which each agent is initially endowed with one house.

distribution, then the agents choose their favorite available object sequentially according to the ordering. RSD is strategy-proof and symmetric (treating agents equally). While the resulting assignment is ex-post efficient, as shown in Bogomolnaia and Moulin (2001) it may entail ex-ante efficiency loss: every agent could strictly prefer another random assignment that first-order stochastically dominates the RSD outcome. A random assignment is then *sd-efficient* if it is not first-order stochastically dominated by another assignment for each agent. They propose a family of *eating mechanisms*, in which agents "eat" the probability shares of objects simultaneously according to their eating speed functions, that characterizes the set of sd-efficient assignments. The central element of the family, the *probabilistic serial mechanism* (PS), gives each agent identical and constant eating speed and satisfies a stronger fairness notion of sd-envy-free. However, while PS outperforms RSD in terms of efficiency and fairness, it is not strategy-proof. An impossibility result from Bogomolnaia and Moulin (2001) shows that if we focus on the class of symmetric mechanisms, then sd-efficiency and strategy-proofness are not compatible. The trade-off and comparison between these two prominent random mechanisms have thus drawn lots of attention in recent studies.

In this paper we further compare these two mechanisms in a more general context where the agents are partitioned into several priority groups. Many practical allocation problems involve such a group structure. For instance, in on-campus dormitory allocation, a school might prioritize students based on seniority, so the fourth year students are placed in the highest priority group, the third year students are in the second priority group and so on. In the presence of group priorities, the fairness requirement is two-fold: within each group the assignment should be symmetric or sd-envy-free, and across groups the priorities should be respected. Naturally, RSD and PS can be implemented in a group-by-group fashion: under group-wise RSD (GRSD) the agents in a higher ranked group draw their ordering lotteries and choose objects before a lower ranked group; similarly, under group-wise PS (GPS), we let the first group "eat" first, the second group "eat" secondly and so on.

In addition to efficiency and within-group fairness, GPS also outperforms GRSD in terms of "respecting priorities" from the ex-ante point of view. For deterministic assignments, respecting priorities is usually referred to the stability concept in the two-sided matching theory (Gale and Shapley, 1962), requiring that there is no *justified-envy*.[2] Adapting this concept to the random environment, *ex-ante stability* (Kesten and Ünver, 2015) is defined by requiring that there is no justified envy for the probability share of any object.

[2]Suppose an agent i envies agent j's assignment, then this envy is "justified" if i has a higher priority than j for j's assignment.

Another approach of generalizing stability to random assignments exploits the fact that, in the deterministic setting, the set of stable assignments is equivalent to the core of the corresponding two-sided matching problem when the priorities are interpreted as preferences of the objects. We define the *strong core* for random assignments and show that it is equivalent to ex-ante stability. Thus ex-ante stability is probably the most relevant stability notion for such a problem. We show that GRSD is only ex-post stable, while GPS is ex-ante stable. Furthermore, any sd-efficient and ex-ante stable assignment can be selected by a group-wise eating mechanism, while any ex-post stable and efficient assignment can be selected by some asymmetric GRSD.

In addition to the axioms discussed above, another common desirable property of matching mechanisms is consistency, which requires that a mechanism should be coherent in choosing assignments for the whole problem and its subproblems.[3] Chambers (2004) provides a natural generalization of the consistency concept to the probabilistic setting: if the random assignment for a subset of agents is realized and they leave the problem with sure objects, then the mechanism should select a random assignment for the reduced problem based on the Bayesian update of the original assignment. However, such a property is too strong that it is not even compatible with symmetry and ex-post efficiency. A natural weaker version of this consistency concept in our context, *Bayesian consistency at the top*, is defined by restricting attention to the departure of some higher ranked priority groups. We show that GRSD is Bayesian consistent at the top, while GPS is not.

One implication of Bayesian consistency at the top is that, if the lotteries for some higher ranked groups are implemented, and the mechanism is reapplied to the reduced problem after the departure of those agents with sure objects, then this two-step procedure should generate the same random assignment from the ex-ante perspective. GPS fails to satisfy this property, and such a non-simultaneous implementation of GPS looses the desirable properties of sd-efficiency, sd-envy-free and ex-ante stability. The reason is that, for those agents in the reduced problem, such a procedure can be considered as a randomization over sd-efficient and ex-ante stable mechanisms. Therefore, both the instability of GRSD and the inconsistency of GPS confirm and extend the old intuition that simply randomizing over "good" deterministic mechanisms might not satisfy good ex-ante properties. On the other hand, in addition to easy implementation and strategy-proofness, the advantage of randomizing over deterministic mechanisms (GRSD) is also demonstrated in its robustness

[3]See Thomson (2011) for a survey on the consistency principle.

to non-simultaneous assignment.

While the axiomatic characterization of RSD is still elusive, PS has been characterized by Bogomolnaia and Heo (2012) and Hashimoto et al. (2014).[4] We generalize the results in Bogomolnaia and Heo (2012) and show that GPS is the only mechanism satisfying sd-efficiency, sd-envy-free, ex-ante stability and bounded invariance, the last of which restricts how a mechanism responses to certain perturbations of preferences.

The random allocation problem with group priorities was first formulated in Che and Kojima (2010). They show that RSD and PS are asymptotically equivalent when the number of copies of each object approaches infinity, and the result still holds if the agents are partitioned into groups and different priority groups are treated differently, such as under GRSD and GPS. In a related study, Manea (2009) shows that the proportion of preference profiles for which RSD is sd-efficient approaches zero as the market becomes large. A similar result can be obtained in our finite market problem: as the group structure becomes more "dense", the sd-efficiency of GRSD is strictly increasing until it is sd-efficient for any preference profile. In the "limit" case, where each priority group has only one agent, both GRSD and GPS are reduced to a simple serial dictatorship. However, the counterpart of the asymptotic result from Che and Kojima (2010) is not true: GRSD and GPS are not monotonically closer to each other as the group structure becomes more dense, which can be actually explained by the intuition of replica economies discussed in Che and Kojima (2010).

1.1 Related literature

This paper is mainly related to two strands of literature: the study of random assignments in pure allocation problems and the study on priority-based allocation (or school choice) problems. The latter starts from the seminal work of Abdulkadiroğlu and Sönmez (2003b). In school choice models assuming strict priorities, random assignments are not needed since the fairness requirement is totally captured by stability. In the context of weak priorities, previous studies had focused on randomizations over deterministic school choice mechanisms by means of a tie-breaking rule, until Kesten and Ünver (2015), who proposed ex-ante stable mechanisms based on the deferred acceptance algorithm of Gale and Shapley (1962).[5] A central issue in school choice is that stability and efficiency are not compatible

[4]Additionally, generalized versions of PS are characterized in the context of multi-unit demands (Heo, 2014) and weak preference domain (Heo and Yılmaz, 2015).

[5]They propose the *fractional deferred acceptance algorithm* and the *fractional deferred acceptance and trading*. Generally these two ex-ante stable mechanisms are not equivalent to GPS in our context.

without restrictions on the priority domains.[6] The problem studied in this paper can be considered as a school choice problem with a homogenous and weak priority structure, for which a stable and efficient deterministic assignment always exists. Thus this simple priority structure enables us to apply the two prominent random allocation mechanisms and analyze both stability and efficiency properties.

The literature on random assignments in pure allocation problems was very small before Bogomolnaia and Moulin (2001). Hylland and Zeckhauser (1979) first study the house allocation problems and proposes the *competitive equilibrium with equal incomes* mechanism. Zhou (1990) shows the incompatibility of efficiency, symmetry and strategy-proofness in a cardinal framework. Abdulkadiroğlu and Sönmez (1998) shows the equivalence of RSD to another random mechanism that selects the core from random endowments. Since its introduction in Bogomolnaia and Moulin (2001), PS has been extended to various contexts, including weak preference domain (Katta and Sethuraman, 2006), allocation with private endowments (Yılmaz, 2009, 2010, Athanassoglou and Sethuraman, 2011) and multi-unit demands (Kojima, 2009). In particular, Yılmaz (2010) proposes a generalized PS mechanism that also characterizes the set of sd-efficient and ex-ante stable assignments, but in the presence of individual property rights.

The rest of the paper is organized as follows. We introduce the model in the next section. Section 3 discusses stability for random assignments. Main results concerning the two competing mechanisms are presented in Section 4, then in Section 5 we characterize GPS and provide comparative statics results. Section 6 concludes. All the proofs are given in Appendix A.1.

2 Preliminaries

Let $\mathcal{N}$ be a set of potential agents and $\mathcal{O}$ a set of potential indivisible objects. Given $N \subseteq \mathcal{N}$ and $O \subseteq \mathcal{O}$, each agent $i \in N$ has a complete, transitive and antisymmetric **preference relation** R_i on O, with P_i denoting its asymmetric component. $U(R_i, a) = \{b \in O : bR_ia\}$ denotes the upper contour set at $a \in O$. Let $\mathcal{R}_O$ be the set of all the preference relations on O. $R = (R_i)_{i \in N} \in \mathcal{R}_O^N$ denotes a **preference profile**. There is a

[6] Ergin (2002) characterizes the priority structures for which stability is compatible with efficiency in the school choice problem with strict priorities. Han (2015) characterizes such priority domain in the case of one-to-one matching with weak priority orderings, and obviously the problem studied in this paper falls into this domain.

partition $Q = \{Q_k\}_{k=1}^q$ of N: $\cup_{k=1}^q Q_k = N$, and $Q_s \cap Q_t = \phi$ if and only if $s \neq t$. So the agents are partitioned into q **priority groups** and we interpret Q_1 as the group with the highest priority. Let $\mathcal{Q}_N$ be the collection of all such partitions of N. A **problem** $\mathcal{E}$ is then summarized as a tuple $\mathcal{E} = \left(N, O, Q = \{Q_k\}_{k=1}^q, R\right)$. For simplicity, we restrict attention to problems with equal number of agents and objects.

Given $\mathcal{E} = \left(N, O, Q = \{Q_k\}_{k=1}^q, R\right)$, an **assignment** is a bistochastic matrix $\pi = [\pi_{ia}]_{i\in N, a\in O}$ such that $\forall i \in N$, $a \in O$, $\pi_{ia} \geqslant 0$, $\sum_{j\in N} \pi_{ja} = \sum_{b\in O} \pi_{ib} = 1$.[7] Each row π_i is the lottery over the objects obtained by agent i; each column π_a specifies the allocation of a to the agents. π is **deterministic** if $\pi_{ia} \in \{0,1\}$ for any $i \in N, a \in O$.[8] Let $\mathcal{A}(\mathcal{E})$ be the set of deterministic assignments of $\mathcal{E}$ and $\Delta\mathcal{A}(\mathcal{E})$ be the set of probability distributions over $\mathcal{A}(\mathcal{E})$. For $p \in \Delta\mathcal{A}(\mathcal{E})$, $S(p) = \{\pi \in \mathcal{A}(\mathcal{E}) : p_\pi > 0\}$ is the support of p. Then $L = \left(\left\{\pi^l\right\}_{l=1}^s; \{p_{\pi^l}\}_{l=1}^s\right)$ is a **lottery representation** of an assignment π if for some $p \in \Delta\mathcal{A}(\mathcal{E})$, $S(p) = \left\{\pi^l\right\}_{l=1}^s$ and $\pi = \sum_{l=1}^s p_{\pi^l}\pi^l$. By the classical *Birkhoff - Von Neumann Theorem* (Birkhoff, 1946, Von Neumann, 1953), every assignment has at least one lottery representation.

A deterministic assignment is efficient if it cannot be Pareto dominated by another deterministic assignment. An assignment is **ex-post efficient** if it can be represented as a lottery over some efficient deterministic assignments. An agent can compare two lotteries over the objects by the first-order stochastic dominance relation R_i^{sd}:

$$\pi_i R_i^{sd} \pi_i' \text{ if } \textstyle\sum_{b\in U(R_i,a)} \pi_{ib} \geqslant \sum_{b\in U(R_i,a)} \pi_{ib}', \ \forall a \in O.$$

Then π is **sd-efficient** if there does not exist π' such that $\pi' \neq \pi$ and $\pi_i' R_i^{sd} \pi_i$, $\forall i \in N$. Sd-efficiency implies ex-post efficiency but not vice versa. In addition to possible ex-ante efficiency loss, a related issue concerning an ex-post efficient assignment is that it might also be represented as a lottery including some inefficient deterministic assignments in the support. In contrast, a sd-efficient assignment can only be represented as lotteries over efficient deterministic assignments.[9]

In our model, the two standard fairness notions are defined within each group: π is **symmetric** if for any $i, j \in Q_k$ for some k, $R_i = R_j$ implies $\pi_i = \pi_j$; π is **sd-envy-free** if

[7]Given $\mathcal{E} = (N, O, Q, R)$, an assignment π and some $S \subseteq N$, we often write $Q|_S, R|_S, \pi|_S$ as the restrictions of these variables with respect to S.

[8]We will abuse the notation slightly for deterministic assignments: denote $\pi(i) = a$ when $\pi_{ia} = 1$.

[9]However, the converse is not true. An assignment might not be sd-efficient even if it can only be represented as lotteries over efficient deterministic assignments. See Abdulkadiroğlu and Sönmez (2003a) for a characterization of sd-efficiency in terms of lottery representations.

for any $i, j \in Q_k$ for some k, $\pi_i R_i^{sd} \pi_j$.

A **mechanism** is a function f that maps each problem $\mathcal{E}$ to an assignment $f(\mathcal{E})$. Then f is said to satisfy a certain property if $f(\mathcal{E})$ satisfies this property for each $\mathcal{E}$. Finally, a mechanism f is **strategy-proof** if for any $\mathcal{E} = (N, O, Q, R)$, $i \in N$ and $R'_i \in \mathcal{R}_O$, we have $f_i(\mathcal{E}) R_i^{sd} f_i(N, O, Q, (R'_i, R_{-i}))$.

3 Stability for random assignments

Symmetry or sd-envy-free requires that the assignment is fair within each group. A notion of respecting the priorities across groups, or stability, needs to be defined in the probabilistic setting. First, a (weak) priority ordering $\succeq_Q$ over the agents can be induced from a partition $Q = \{Q_k\}_{k=1}^q \in \mathcal{Q}_N$: for any $i, j \in N$, $i \succeq_Q j$ if $i \in Q_s, j \in Q_t, s \leqslant t$. Let $\succ_Q$ be the asymmetric part of $\succeq_Q$. Then the stability concept for deterministic assignments is standard: for any $\mathcal{E} = (N, O, Q, R)$, $\pi \in \mathcal{A}(\mathcal{E})$ is **stable** if for any $i, j \in N$, $\pi(i) P_j \pi(j)$ implies $i \succeq_Q j$. A weak form of stability for random assignments is **ex-post stability**: an assignment is ex-post stable if it can be represented as a lottery over some stable deterministic assignments. Ex-post stability has similar issues to those of ex-post efficiency: an ex-post stable assignment may not be fair from the ex-ante point of view when agents compare their lotteries, and it might be represented as a lottery including some unstable assignments in the support.

Example 1. $N = \{1, 2, 3, 4, 5\}$, $O = \{a, b, c, d, e\}$, $Q_1 = \{1, 2, 3, 4\}$, $Q_2 = \{5\}$. Consider the following preferences R and assignment π:

$R_1:$	a	b	c	d	e
$R_2:$	a	b	c	d	e
$R_3:$	b	d	a	c	e
$R_4:$	a	e	b	c	d
$R_5:$	c	e	a	b	d

π	a	b	c	d	e
1	$\frac{1}{3}$	$\frac{5}{24}$	$\frac{3}{8}$	$\frac{1}{12}$	0
2	$\frac{1}{3}$	$\frac{5}{24}$	$\frac{3}{8}$	$\frac{1}{12}$	0
3	0	$\frac{7}{12}$	0	$\frac{5}{12}$	0
4	$\frac{1}{3}$	0	0	0	$\frac{2}{3}$
5	0	0	$\frac{1}{4}$	$\frac{5}{12}$	$\frac{1}{3}$

π can be represented as the following lottery:[10]

$$L = (abdec, acbed, badec, cabed, cdbae, dcbae, bcdae, cbdae; \tfrac{1}{8}, \tfrac{5}{24}, \tfrac{1}{8}, \tfrac{5}{24}, \tfrac{1}{12}, \tfrac{1}{12}, \tfrac{1}{12}, \tfrac{1}{12})$$

[10]We use "*abcde*" to denote the deterministic assignment in which $\pi(1) = a, \pi(2) = b$ and so on.

where each deterministic assignment in the support of L is stable, so π is ex-post stable. However, there exists another lottery representation of π:

$$L' = (dabec, abdec, acbed, badec, cabed, cdbae, bcdae, cbdae; \tfrac{1}{12}, \tfrac{1}{8}, \tfrac{5}{24}, \tfrac{1}{24}, \tfrac{5}{24}, \tfrac{1}{12}, \tfrac{1}{6}, \tfrac{1}{12})$$

With probability $\frac{1}{12}$ the deterministic assignment "*dabec*" is chosen, which is not stable since $1 \succ_Q 5$ and cP_1d.

Agent 1 in the example might envy agent 5's probability share of object c, since 1 is ranked higher than 5 and is assigned a worse object (d) with a positive probability. In this sense such an assignment is not fair ex-ante. A direct generalization of the stability concept to the probabilistic setting rules out such violations of priority rights:

Definition 1. Given $\mathcal{E} = (N, O, Q, R)$, an assignment π is **ex-ante stable** if for any $i, j \in N$ and $a, b \in O$ such that $\pi_{ia} > 0, \pi_{jb} > 0$, aP_jb implies $i \succeq_Q j$.

Kesten and Ünver (2015) define the same concept for the school choice problems, and they show that even in this broader class of problems where the priority ordering $\succeq_Q$ varies across objects (schools), an ex-ante stable assignment always exists.[11] Ex-ante stability implies ex-post stability, and an ex-ante stable assignment can only be represented as lotteries over stable deterministic assignments.[12]

In the deterministic setting, the set of stable assignments for a general priority-based allocation problem is equivalent to the core of the corresponding two-sided matching problem when the priorities of the objects are interpreted as their preferences. Hence another natural approach of generalizing stability to random assignments from an ex-ante perspective is to define a "probabilistic core". Intuitively, a core for random assignments also has the interpretation of respecting priorities, since given an assignment not in the core, there exists some coalition within which objects can be reallocated to higher ranked agents to

[11] In particular, when each school a is endowed with a priority ordering $\succeq_a$, they define an assignment π as ex-ante stable if for any $i, j \in N$ and $a, b \in O$ such that $\pi_{ia} > 0, \pi_{jb} > 0$, aP_jb implies $i \succeq_a j$.

[12] However, similar to the case of sd-efficiency, an assignment may not be ex-ante stable even if it can only be represented as lotteries over stable deterministic assignments. The reason is also essentially the same, as discussed in Abdulkadiroğlu and Sönmez (2003a): although an assignment can only be represented as lotteries over stable (efficient) deterministic assignments, such an assignment could also be induced by a lottery including some infeasible and unstable (inefficient) deterministic assignments in the support. Using the terminology of Abdulkadiroğlu and Sönmez (2003a), it can be easily seen that an assignment is ex-ante stable if and only if given any lottery representation L, any set of (feasible or infeasible) deterministic assignments that are *frequency equivalent* to the assignments in the support of L are stable.

make them better-off. Let the first-order stochastic dominance relation $\succeq_Q^{sd}$ for an object be defined in the same way as R_i^{sd}.

Definition 2. Given $\mathcal{E} = (N, O, Q, R)$, a coalition $C \subseteq N \cup O$ **blocks** an assignment π if $|C \cap N| = |C \cap O|$, and there exists another assignment π' such that *(i)* $\sum_{a \in C \cap O} \pi'_{ia} = 1$, $\forall i \in C \cap N$, and $\sum_{i \in C \cap N} \pi'_{ia} = 1$, $\forall a \in C \cap O$; *(ii)* there does not exist $i \in C \cap N$ such that $\pi_i R_i^{sd} \pi'_i$, or $a \in C \cap O$ such that $\pi_a \succeq_Q^{sd} \pi'_a$. An assignment is in the **strong core** of $\mathcal{E}$ if it is not blocked by any coalition.[13]

A coalition C can block an assignment π if by distributing probability shares among themselves, each member $c \in C$ can obtain a lottery that is not first-order stochastically dominated by π_c. Such a core concept was first defined by Manjunath (2013) for the marriage problem, and it is strong since it is a subset of a core defined with respect to any profile of expected utility functions that are consistent with the ordinal rankings of the agents and the objects. A weaker core concept can be defined by requiring that a coalition C blocks an assignment π if each member $c \in C$ can obtain a (different) lottery that first-order stochastically dominates π_c. But an assignment in such a weak core may not be even ex-post stable.[14] The strong core further justifies the notion of ex-ante stability.

Proposition 1. *For any $\mathcal{E}$, an assignment is ex-ante stable if and only if it is in the strong core of $\mathcal{E}$.*

The equivalence closely hinges on the fact that the priority structure is homogeneous and does not hold for a general priority-based allocation problem. In fact, while an ex-ante stable assignment always exists, as shown in Manjunath (2013) the strong core is not necessarily non-empty if the priority orderings vary across objects.

4. Two competing random mechanisms

To respect the group priorities, the *random serial dictatorship* (RSD) can be simply implemented in a group-by-group fashion: pick from the uniform distribution an ordering of

[13] A blocking coalition can contain an arbitrary number of agents and objects if we allow each agent and object to be self-assigned for some probability, and all the following results are unchanged.

[14] The following simple example illustrates this point. Suppose $N = \{1, 2, 3\}, O = \{a, b, c\}, Q_1 = \{1, 2\}, Q_2 = \{3\}$, and preferences are given by $R_1 : c, a, b; R_2 : c, a, b; R_3 : a, b, c$. Consider the assignment π, where $\pi_{1b} = \pi_{1c} = \pi_{2a} = \pi_{2c} = \pi_{3a} = \pi_{3b} = 0.5$. π has a unique lottery representation $L = (bca, cab; 0.5, 0.5)$ and it is not ex-post stable since the first assignment in the support of L is not stable. However, it can be easily verified that π is in the weak core.

those agents in the highest ranked group, then let them choose their favorite available object sequentially according to the ordering; after they leave the problem with their assignments, repeat this process iteratively for the reduced problem. From an ex-ante perspective, such a procedure is equivalent to assigning equal probabilities to all the serial dictatorships that are consistent with the group priorities. Specifically, given $\mathcal{E} = (N, O, Q, R)$ and an ordering of the agents θ ($\theta : N \to \{1, 2, ..., |N|\}$, θ is a bijection), let the deterministic assignment resulting from the serial dictatorship be $f^{\theta}(\mathcal{E})$.[15] Let $\Theta(Q) = \{\theta : \theta(i) < \theta(j)$ *if* $i \succ_Q j, \forall i, j \in N\}$ be the set of orderings consistent with the group priorities. Then the **group-wise random serial dictatorship** (GRSD) assignment is defined as

$$f^{GRSD}(\mathcal{E} = (N, O, Q, R)) = \frac{1}{|\Theta(Q)|} \sum_{\theta \in \Theta(Q)} f^{\theta}(\mathcal{E}).$$

On the other hand, an *eating algorithm* is defined with respect to a profile of eating speed functions on the unit time interval. Each agent "eats" the probability shares of available objects according to her speed function and in the order of her preference list. Formally, ω_i is an eating speed function for $i \in \mathcal{N}$ if $\omega_i(t) : [0, 1] \to \mathbf{R}_+$, ω_i is measurable and $\int_0^1 \omega_i(t)dt = 1$. Let $\mathcal{W}$ be the set of eating speed functions. Given $\mathcal{E} = (N, O, Q, R)$ and $\omega = (\omega_i)_{i \in N} \in \mathcal{W}^N$, the eating algorithm f^{ω} is defined as follows. For $a \in O' \subseteq O$, let $M(a, O') = \{i \in N : aP_ib, \forall b \in O' \setminus \{a\}\}$. Let $O^0 = O, t^0 = 0, \pi^0 = [0]$. Given $O^0, t^0, \pi^0, ..., O^{k-1}, t^{k-1}, \pi^{k-1}$, for any $a \in O^{k-1}$ define

(4.1) $t^k_a = min\left\{t : \sum_{i \in M(a, O^{k-1})} \int_{t^{k-1}}^{t} \omega_i(t)dt + \sum_{i \in N} \pi^{k-1}_{ia} = 1\right\}$, $t^k_a = 1$ if $M(a, O^{k-1}) = \phi$,
(4.2) $t^k = min_{a \in O^{k-1}} t^k_a$,
(4.3) $O^k = \left\{a \in O^{k-1} : t^k_a > t^k\right\}$,
(4.4) $\pi^k_{ia} = \pi^{k-1}_{ia} + \int_{t^{k-1}}^{t^k} \omega_i(t)dt$ if $i \in M(a, O^{k-1}), \pi^k_{ia} = \pi^{k-1}_{ia}$ otherwise.

The algorithm terminates when for some $k', O^{k'} = \phi$. Then $f^{\omega}(\mathcal{E}) = \pi^{k'}$. f is an **eating mechanism** with respect to $\omega \in \mathcal{W}^{\mathcal{N}}$ if for any $\mathcal{E} = (N, O, Q, R)$, $f(\mathcal{E}) = f^{\omega'}(\mathcal{E})$, where $\omega'_i = \omega_i$ for all $i \in N$. The central element of the family of eating mechanisms is the **probabilistic serial mechanism** (PS), for which $\omega_i(t) = 1$ for all $t \in [0, 1]$ and $i \in \mathcal{N}$.

Remark 1. Heo (2014) introduced the idea of a *consumption process* such that any random

[15] In particular, agent $\theta^{-1}(1)$ picks her favorite object first, then agent $\theta^{-1}(2)$ picks her favorite object from the remaining ones and so on.

assignment can be considered to be generated by a process similar to PS.[16] This notion can be generalized by allowing any profile of eating speed functions in the consumption process, which will play an important role in our proof of the results concerning eating mechanisms (Proposition 2 and 5). Given $\mathcal{E} = (N, O, Q, R)$, $m(t) = (m_i(t))_{i \in N}$, where $m_i(t) : [0,1] \to O$ for each $i \in N$, is a **consumption schedule** if $m_i(t')R_i m_i(t'')$ for any $0 \leqslant t' \leqslant t'' \leqslant 1$ and $i \in N$. $m(t)$ **represents** an assignment π at $\omega \in \mathcal{W}^N$ if $\int_{t \in m_i^{-1}(a)} \omega_i(t)dt = \pi_{ia}$ for any $i \in N, a \in O$, and $m_i(t)$ is available at t for all i and t such that i has not finished consuming at t, i.e., $\sum_{j \in N} \int_{s \in m_j^{-1}(m_i(t)) \cap [0,t]} \omega_j(s)ds < 1$ if $\int_{s=0}^{t} \omega_i(s)ds < 1$. Given any $\omega \in \mathcal{W}^N$, the continuity of $F_{\omega_i}(t) = \int_0^t \omega_i(s)ds$ on $[0,1]$ implies that any assignment can be represented by some consumption schedule at ω. The defining feature of a consumption schedule representing an eating algorithm outcome is that each agent is always consuming the best available object:

Claim 1. *$m(t)$ represents $f^{\omega}(\mathcal{E})$ at ω if and only if for each i and t such that $F_{\omega_i}(t+\epsilon) > F_{\omega_i}(t)$ for any $\epsilon > 0$, $m_i(t)$ is the best available object for i at t.*

Claim 1 follows immediately from the construction of eating algorithms. Using (4.1)-(4.4), a consumption schedule representing $f^{\omega}(\mathcal{E})$ at ω can be constructed as follows. Let $\underline{k}(t) = max\ \left\{k : t^k \leqslant t\right\}$. For any $i \in N$ define

(4.5) $m_i(t) = a$ if $i \in M(a, O^{\underline{k}(t)})$ and $\sum_{b \in O} \pi_{ib}^{\underline{k}(t)} < 1$, $m_i(t) = m_i(t^{\underline{k}(t)-1})$ if $\sum_{b \in O} \pi_{ib}^{\underline{k}(t)} = 1$

Given $N \subseteq \mathcal{N}$ and $Q = \{Q_k\}_{k=1}^{q} \in \mathcal{Q}_N$, to adapt to the priority structure we let agents eat group-by-group, i.e., for each $i \in N$, $i \in Q_k$ implies that $\omega_i(t) = 0$ for $t \in [0,1] \backslash \left[\frac{k-1}{q}, \frac{k}{q}\right]$. Denote the set of such eating speed function profiles as $\mathcal{W}(Q)$. Then f is a **group-wise eating mechanism** (GE) if for any $N \subseteq \mathcal{N}$ and $Q \in \mathcal{Q}_N$, there exists $\omega \in \mathcal{W}(Q)$ such that $f(\mathcal{E} = (N, O, Q, R)) = f^{\omega}(\mathcal{E})$ for all $O \subseteq \mathcal{O}$ and $R \in \mathcal{R}_O^N$.

If for any $\mathcal{E} = (N, O, Q = \{Q_k\}_{k=1}^{q}, R)$, $f(\mathcal{E}) = f^{\omega}(\mathcal{E})$, where $\omega \in \mathcal{W}(Q)$ and $\omega_i(t) = q$ for all $i \in N$ and t such that $\omega_i(t) > 0$, then f is the **group-wise probabilistic serial mechanism** (GPS), denoted as f^{GPS}.

4.1 Stability and efficiency

[16] This representation is the key to various characterizations of PS, see Bogomolnaia and Heo (2012), Heo (2014) and Heo and Yılmaz (2015).

Properties of the random serial dictatorship and eating mechanisms carry over to their group-wise versions: GRSD is symmetric, ex-post efficient and strategy-proof; while GEs are sd-efficient and GPS is sd-envy-free, generally they are not strategy-proof.

It can be easily seen that given a problem $\mathcal{E}$, any GE is ex-ante stable, since given any two agents i and j with $i \succ_Q j$, any object better than i's last choice in the support of her lottery must be exhausted before j's group starts to eat. GRSD is ex-post stable, since for any $\theta \in \Theta(Q)$, $f^\theta(\mathcal{E})$ is stable. However, GRSD may not be ex-ante stable.

Example 1. *(continued)* As discussed before, π is not ex-ante stable, and in fact it is the GRSD outcome. GPS yields the following ex-ante stable assignment π':

π'	a	b	c	d	e
1	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{4}{9}$	0	0
2	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{4}{9}$	0	0
3	0	$\frac{5}{9}$	0	$\frac{4}{9}$	0
4	$\frac{1}{3}$	0	0	0	$\frac{2}{3}$
5	0	0	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{1}{3}$

By adding group priorities into the model and considering ex-post and ex-ante stability, the comparison between the two mechanisms is parallel to that in the pure allocation case. While eating mechanisms characterize the set of sd-efficient assignments, GEs characterize the set of sd-efficient and ex-ante stable assignments.

Proposition 2. *Given $\mathcal{E} = (N, O, Q, R)$, $f^\omega(\mathcal{E})$ is sd-efficient and ex-ante stable for any $\omega \in \mathcal{W}(Q)$; for any sd-efficient and ex-ante stable π of $\mathcal{E}$, there exists $\omega \in \mathcal{W}(Q)$ such that $f^\omega(\mathcal{E}) = \pi$.*

Remark 2. To illustrate the basic idea of the proof for the second statement, consider the simple case of $q = 2$. Sd-efficiency of π implies that there exists $\omega \in \mathcal{W}^N$ such that $f^\omega(\mathcal{E}) = \pi$. Without loss of generality it can be assumed that $\omega_i(t) = 0$ for all $t \in \left(\frac{1}{2}, 1\right]$ and $i \in N$. Let ω' be such that $\omega'_i(t) = \omega_i(t)$ for $i \in Q_1$ and $\omega'_i(t) = \omega_i(t - \frac{1}{2})$ for $i \in Q_2$, i.e., the second group is separated and moved to the second half of the time interval to eat. Then $\omega' \in \mathcal{W}(Q)$ and it is sufficient to show that $f^\omega(\mathcal{E}) = f^{\omega'}(\mathcal{E})$. Ex-ante stability can be invoked to show that the assignments are the same for the first group. To show that the assignment is also unchanged for the second group, it is in fact equivalent to showing that any eating mechanism is *consistent*. Consistency is a concept of robustness and requires that a mechanism is coherent in choosing assignments for the whole problem and its subproblems. In formally defining consistency we allow fractional endowments and suppress the partitions on agents: let $N \subseteq \mathcal{N}$ and $I = (I_a)_{a\in\mathcal{O}}$ be an initial endowment vector with $I_a \in [0,1]$ for all $a \in \mathcal{O}$ and $\sum_{a\in\mathcal{O}} I_a = |N|$; let $R \in \mathcal{R}^N_{\mathcal{O}}$, then $e = (N, I, R)$ is an allocation problem for which an assignment π is a $|N| \times |\mathcal{O}|$ stochastic matrix with $\sum_{i\in N} \pi_{ia} = I_a$ for each $a \in \mathcal{O}$. The reduced problem of e with respect to an assignment π and a subset of agents $S \subseteq N$ is $r^{\pi}_S(e) = (S, I' = \sum_{i\in S} \pi_i, R|_S)$. An **extended mechanism** $\bar{f}$ maps each problem e to an assignment. Then $\bar{f}$ is **consistent** if $\bar{f}(e)|_S = \bar{f}(r^{\bar{f}(e)}_S(e))$ for any $e = (N, I, R)$ and $S \subseteq N$. In the appendix A.1 we show that any eating mechanism is consistent, as an intermediate step of the proof for Proposition 2.[17]

A similar result can be established for a generalized family of random serial dictatorships. The counterpart of a general GE is an asymmetric GRSD, in which an ordering consistent with the group priorities is picked from an arbitrary distribution. Formally, given $\mathcal{E} = (N, O, Q, R)$ and $p \in \Delta\Theta(Q)$, let $f^p(\mathcal{E}) = \sum_{\theta\in\Theta(Q)} p_\theta f^\theta(\mathcal{E})$. Then f is an **asymmetric GRSD** if for any $N \subseteq \mathcal{N}$ and $Q \in \mathcal{Q}_N$, there exists $p \in \Delta\Theta(Q)$ such that $f(\mathcal{E} = (N, O, Q, R)) = f^p(\mathcal{E})$ for all $O \subseteq \mathcal{O}$ and $R \in \mathcal{R}^N_O$.

Define an assignment as **ex-post stable & efficient** if it can be represented as a lottery over some deterministic assignments, each of which is both stable and efficient.

Proposition 3. *Given $\mathcal{E} = (N, O, Q, R)$, $f^p(\mathcal{E})$ is ex-post stable & efficient for any $p \in \Delta\Theta(Q)$; for any ex-post stable & efficient π, there exists $p \in \Delta\Theta(Q)$ such that $f^p(\mathcal{E}) = \pi$.*

Thus the family of asymmetric GRSDs characterizes the set of ex-post stable & efficient assignments. An ex-post stable & efficient assignment is obviously ex-post stable and ex-post efficient, but is the converse true? or, equivalently, do GRSDs characterize the set of ex-post stable and ex-post efficient assignments? The following example shows that the answer is no.

Example 2. $N = \{1, 2, 3, 4, 5\}$, $O = \{a, b, c, d, e\}$, $q = 2$, $Q_1 = \{1, 2, 3, 4\}$, $Q_2 = \{5\}$. Consider the following preference profile and an assignment π:

[17] Thomson (2010) shows the consistency of PS, Heo (2014) shows the consistency of a generalized PS defined for allocation problems with multi-unit demands.

$R_1: \; a \; b \; c \; e$
$R_2: \; a \; e \; b \; c$
$R_3: \; d \; c \; e$
$R_4: \; b \; d \; e \; c$
$R_5: \; e \; c$

π	a	b	c	d	e
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
2	$\frac{1}{2}$	$\frac{1}{2}$	0	0	0
3	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0
4	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
5	0	0	$\frac{1}{2}$	0	$\frac{1}{2}$

Two lottery representations of π are given as follows:

$$L = (abcde, badec; \tfrac{1}{2}, \tfrac{1}{2}),$$
$$L' = (abdec, bacde; \tfrac{1}{2}, \tfrac{1}{2}).$$

The two deterministic assignments in L are efficient, while the two deterministic assignments in L' are stable. Thus π is ex-post efficient and ex-post stable. However, π is not ex-post stable & efficient. Consider any lottery representation of π, there exists some deterministic assignment π' in the support such that $\pi'(2) = b$. If π' is efficient, then $\pi'(4) = d$. Since $\pi_{3d} = \pi_{3c} = \pi_{5c} = \pi_{5e} = \frac{1}{2}$, we have $\pi'(3) = c$ and $\pi'(5) = e$. But eP_2b and $2 \succ_Q 5$, so π' is not stable. Hence, there does not exist a lottery representation of π such that each deterministic assignment in the support is both stable and efficient.

4.2 Non-simultaneous assignment

In Remark 2 we briefly discussed the consistency principle by imaging the departure of a subset of agents with their probabilistic assignments. A consistent mechanism recommends the same assignments for the remaining agents when it is "reapplied" to the reduced problem. However, it is more realistic to consider the situation where a subset of agents' lotteries are implemented and they leave the problem with sure objects. A natural requirement of the consistency principle is that the mechanism should recommend an assignment for the reduced problem based on the Bayesian update of the original assignment, conditional on the deterministic assignment of the leaving agents. This is the consistency concept proposed by Chambers (2004). The Bayesian update requires us to work with random assignments in terms of probability distributions over deterministic assignments, rather than bistochastic matrices, thus we have to "decompose" a mechanism: $\widetilde{f}$ is a **decomposition** of a mechanism f if $\widetilde{f}$ maps each $\mathcal{E}$ to an element in $\Delta\mathcal{A}(\mathcal{E})$ and for each $\mathcal{E}$, $\sum_{\pi\in\mathcal{A}(\mathcal{E})} \widetilde{f}_\pi(\mathcal{E})\pi = f(\mathcal{E})$. Adapting the consistency concept from Chambers (2004) to our context, we say a mechanism f is **Bayesian consistent** if given some decomposition $\widetilde{f}$ of

f, for any $\mathcal{E} = (N, O, Q, R)$, $S \subseteq N$ and $\pi' \in \mathcal{A}(S, O, Q|_S, R|_S)$ such that there exists some $\pi'' \in \mathcal{A}(\mathcal{E})$ with $\widetilde{f}_{\pi''}(\mathcal{E}) > 0$ and $\pi''|_S = \pi'$, we have

$$\widetilde{f}_{\pi|_{N\setminus S}}(N \setminus S, O \setminus \{\pi'(S)\}, Q|_{N\setminus S}, R|_{N\setminus S}) = \frac{\widetilde{f}_{\pi}(\mathcal{E})}{\sum_{\bar{\pi}\in\mathcal{A}(\mathcal{E}),\bar{\pi}|_S=\pi'} \widetilde{f}_{\bar{\pi}}(\mathcal{E})}$$

for any $\pi \in \mathcal{A}(\mathcal{E}), \pi|_S = \pi'$.

Bayesian consistency is a strong requirement and Chambers (2004) shows that the only mechanism that satisfies Bayesian consistency and symmetry is the uniform rule, i.e., each agent is assigned each object with equal probability. A natural way to weaken this property in our model is to only consider the departure of those higher ranked groups of agents, who might also be given the privilege of obtaining objects earlier. Then **Bayesian consistency at the top** is defined by restricting S to be $\cup_{k=1}^{n} Q_k$ for some $1 \leqslant n \leqslant q$, for any $\mathcal{E} = \left(N, O, Q = \{Q_k\}_{k=1}^{q}, R\right)$, in the definition of Bayesian consistency.

Proposition 4. *f^{GRSD} is Bayesian consistent at the top; f^{GPS} is not Bayesian consistent at the top.*

f^{GRSD} can be naturally decomposed to assign equal probability to all the admissible serial dictatorships, and its Bayesian consistency at the top follows from the fact that it can be implemented by assigning sure objects to agents group-by-group, as mentioned at the beginning of section 4. To see that f^{GPS} is not coherent in its assignments when certain groups at the top leave the problem with sure objects, consider the following property of a mechanism f that satisfies Bayesian consistency at the top. For any $\mathcal{E} = \left(N, O, Q = \{Q_k\}_{k=1}^{q}, R\right)$, $S = \cup_{k=1}^{n} Q_k$ for some $1 \leqslant n \leqslant q$, there exists a lottery representation $(\pi^1, \pi^2, ..., \pi^k; p_{\pi^1}, p_{\pi^2}, ..., p_{\pi^k})$ of $f(\mathcal{E})|_S$ such that if this lottery is implemented, the agents in S leave the problem with sure objects and f is reapplied to the reduced problem, then this procedure generates the same random assignment from the ex-ante point of view, i.e.,

$$\sum_{l=1}^{k} p_{\pi^l} f(N \setminus S, O \setminus \pi^l(S), Q|_{N\setminus S}, R|_{N\setminus S}) = f(\mathcal{E})|_{N\setminus S}$$

f^{GPS} does not even satisfy this weaker requirement of robustness, which is defined on a mechanism directly and does not involve decompositions of a mechanism. In fact, applying to GPS, the procedure described above might yield an assignment that is not sd-efficient, ex-ante stable or sd-envy-free. Intuitively, the reason is that for those agents in the reduced problem, such a procedure can be considered as a convex combination of sd-efficient and ex-ante stable mechanisms, and as we have seen from the case of GRSD, randomizing over (sd-) efficient and (ex-ante) stable mechanisms can lead to lack of efficiency and stability ex-ante. Therefore, while GPS outperforms GRSD in terms of efficiency and fairness (within and across groups), the advantages of GPS disappear in situations where some higher ranked groups have to be assigned sure objects first before lotteries are assigned to the remaining agents. In such cases, GRSD might be a preferred mechanism due to its easy implementation and robustness to non-simultaneous assignment, as well as strategy-proofness.

In sum, our analysis in this section extends the trade-off between two approaches of generating random assignments: the first approach randomizes over deterministic mechanisms, while the second approach treats the probability share of each object as a perfectly divisible good to be allocated. Loss of ex-ante efficiency and fairness from the first approach are further demonstrated by the ex-ante instability of GRSD and Bayesian inconsistency of GPS, while the advantage of the first approach can also be seen from its robustness to non-simultaneous assignment.

5. Discussion

5.1 A characterization of GPS

In the allocation problems without priorities, Bogomolnaia and Heo (2012) show that the probabilistic serial mechanism is the only mechanism that satisfies sd-efficiency, sd-envy-free and *bounded invariance*. Bounded invariance requires that if an agent changes her preferences but her preference ranking over the upper contour set of some object a remains the same, then every agent's share of a does not change. Formally, given $\mathcal{E} = (N, O, Q, R)$, $i \in N$ and $a \in O$, let $R_i(a)$ be the relation $R_i \in \mathcal{R}_O$ restricted to $U(R_i, a)$. Then f satisfies **bounded invariance** if for any $\mathcal{E} = (N, O, Q, R)$, $i \in N$, $a \in O$ and $R'_i \in \mathcal{R}_O$ such that $R_i(a) = R'_i(a)$, $f_{ja}(\mathcal{E}) = f_{ja}(N, O, Q, (R'_i, R_{-i}))$ for all $j \in N$.

In our model, sd-envy-free is only required within each group. By complementing sd-envy-free with ex-ante stability, a simple generalization of the characterization in Bogomolnaia and Heo (2012) can be obtained:

Proposition 5. *f^{GPS} is the only mechanism that satisfies sd-efficiency, sd-envy-free, bounded invariance and ex-ante stability.*

We show that the axioms are independent. Bogomolnaia and Heo (2012) gives a mechanism that satisfies sd-efficiency and sd-envy-free but not bounded variance for allocation problems without priorities. In our context, applying such a mechanism to the last group and GPS to the other groups yields a mechanism that satisfies all the axioms except bounded invariance. Similarly, applying the uniform rule to the last group and GPS to the other groups gives a mechanism that satisfies all the axioms except sd-efficiency. Serial dictatorships consistent with the group priorities satisfy all the axioms except sd-envy-free. Finally, PS satisfies all the axioms except ex-ante stability.

5.2 Comparative statics

Given $\mathcal{E} = \left(N, O, \{Q_k\}_{k=1}^{q}, R\right)$, if $q = 1$, then this is a standard house allocation problem and there is a trade-off between (G)RSD and (G)PS in terms of sd-efficiency, sd-envy-free and strategy-proofness. If $q = |N|$, the problem is perfectly hierarchical. In this case both GRSD and GPS are reduced to a simple serial dictatorship which satisfies all the desiderata. We are interested in possible monotonic properties concerning GRSD and GPS as q increases from 1 to $|N|$, i.e., when the priority structure becomes more "dense". For house allocation problems, Manea (2009) shows that as the number of agents and objects becomes large, the proportion of preference profiles for which RSD is sd-efficient vanishes. Che and Kojima (2010) show that as the number of copies of each object type goes to infinity, RSD and PS are asymptotically equivalent. Hence there are two natural conjectures corresponding to the two asymptotic results: as q increases, (i) the sd-efficiency of GRSD is increasing; (ii) GRSD and GPS are monotonically closer to each other.

Fix N and O. A partition $Q' = \{Q'_k\}_{k=1}^{q'} \in \mathcal{Q}_N$ is a **refinement** of the partition $Q = \{Q_k\}_{k=1}^{q} \in \mathcal{Q}_N$ if $q' > q$ and $i \succ_Q j$ implies $i \succ_{Q'} j$ for all $i, j \in N$. For each $Q \in \mathcal{Q}_N$, define $\mathit{eff}(Q)$ as the proportion of preference profiles such that the GRSD assignment is sd-efficient:

$$\mathit{eff}\,(Q) = \frac{\left|\left\{R \in \mathcal{R}_O^N : f^{GRSD}(N, O, Q, R) \text{ is sd-efficient}\right\}\right|}{\left|\mathcal{R}_O^N\right|}$$

Now we state and show that the first conjecture is true.

Proposition 6. *Q' is a refinement of Q implies $\mathit{eff}\,(Q') \geqslant \mathit{eff}\,(Q)$, with strict inequality if $\mathit{eff}\,(Q) \neq 1$.*

Therefore, as the priority structure becomes more dense, the sd-efficiency of GRSD is strictly increasing until it is sd-efficient for any preference profile. Intuitively, as the agents are separated into more priority groups, the GRSD assignment becomes "less uncertain", thus the chances of improving upon the GRSD outcome through Pareto improvement exchanges of probability shares become smaller. However, the following example shows that the second conjecture is not true: although in the "limit" GRSD coincides with GPS, generally the difference between these two mechanisms can become larger as the priority structure becomes more dense.

Example 3. $N = \{1,2,3,4,5,6\}$, $O = \{a,b,c,a',b',c'\}$, $Q_1 = N$, $Q'_1 = \{1,3,5\}$, $Q'_2 = \{2,4,6\}$. R is given as follows:

$$\begin{array}{lcccccc} R_1: & a & b & c & a' & b' & c' \\ R_2: & a & b & c & a' & b' & c' \\ R_3: & a & c & b & a' & c' & b' \\ R_4: & a & c & b & a' & c' & b' \\ R_5: & b & c & a & b' & c' & a' \\ R_6: & b & c & a & b' & c' & a' \end{array}$$

We consider three usual metrics on assignments:

$$\|\pi - \pi'\|_1 = sup_{\ i\in N, a\in O}\ |\pi_{ia} - \pi'_{ia}|,$$
$$\|\pi - \pi'\|_2 = \textstyle\sum_{i\in N, a\in O} |\pi_{ia} - \pi'_{ia}|,$$
$$\|\pi - \pi'\|_3 = \sqrt{\textstyle\sum_{i\in N, a\in O} (\pi_{ia} - \pi'_{ia})^2}.$$

Detailed assignments from GRSD and GPS as well as the distances between them are given in Appendix A.2. Although Q' is a refinement of Q, the gap between GRSD and GPS is larger under Q', with respect to any of the three metrics.

The reason behind the increasing gap between GPS and GRSD as q increases in this example is of more interest. The house allocation problem (N, O, Q, R) here is similar to a replicated economy, where the base economy consists of the agents $\{1,3,5\}$ and the objects $\{a,b,c\}$. Che and Kojima (2010) show that, as a special case of their model, when a base economy is replicated enough times, and thus the number of copies of each object as well as the number of agents with identical preferences as someone in the base economy becomes large, the difference between RSD and PS approaches zero. In Example 3, it can be expected that, fixing $q = 1$, the difference between (G)RSD and (G)PS becomes even smaller as we replicate the base economy more times. However, once we put each "copy" of the agents in a separate priority group (as under Q'), the difference between GRSD and GPS in the base economy is replicated, thus the gap between the two mechanisms is enlarged under the refined priority structure.

6. Conclusion

We conclude by discussing some possible extensions. First, it has been assumed that $|N| = |O|$ in any problem throughout this paper, but it is easy to check that all the results still hold if there are different numbers of agents and objects, agents and objects can be self-assigned for some probabilities or there are multiple copies of each object. This allocation problem with group priorities is commonly seen in practice and simple enough such that RSD and PS can be easily adapted to. Yılmaz (2010) defines a family of generalized eating mechanisms that characterizes the set of sd-efficient and individually rational assignments for the problem of *house allocation with existing tenants* (Abdulkadiroğlu and Sönmez, 1999). The individual rationality condition is equivalent to ex-ante stability if the problem is interpreted as priority-based allocation, so his result is the counterpart of Proposition 2 for the priority structures where for some objects a single agent (the existing tenant) is ranked higher than others. Therefore, an interesting question is whether the eating mechanisms can be generalized to more complex weak priority structures such that they still characterize the set of sd-efficient and ex-ante stable assignments, while satisfying some additional fairness requirement that is not captured by ex-ante stability.

Appendix A.1 Proofs

Proof of Proposition 1. Consider any $\mathcal{E} = (N, O, Q, R)$. Suppose π is not ex-ante stable, then there exist $i, j \in N$ and $a, b \in O$ such that $\pi_{ia} > 0, \pi_{jb} > 0$, $aP_j b$, and $j \succ_Q i$. Let π' be any assignment with $\pi'_{ja} = 1$, then the coalition $C = \{a, j\}$ blocks π, thus π is not in the strong core of $\mathcal{E}$. For the other direction, suppose π is ex-ante stable, but assume to the contrary it is not in the strong core of $\mathcal{E}$. Then there exists a coalition C and an assignment π' such that C blocks π. Given any $i \in C \cap N$, since it is not that $\pi_i R_i^{sd} \pi'_i$, there exist $a \in C \cap O$ and $b \in O$ such that $\pi'_{ia} > 0, \pi_{ib} > 0$ and $aP_i b$. In this case denote $i\varphi a$. Similarly, for any $a \in C \cap O$, there exist $i \in C \cap N$ and $j \in N$ such that $\pi'_{ia} > 0, \pi_{ja} > 0$ and $i \succ_Q j$. In this case denote $a\varphi i$. φ is asymmetric: if $i\varphi a$ and $a\varphi i$, then there exist $b \in O, j \in N$ such that $aP_i b, \pi_{ib} > 0$ and $i \succ_Q j, \pi_{ja} > 0$, contradicting to π being ex-ante stable. Then by the definition and asymmetry of φ, $i\varphi a\varphi j$ implies $j \succ_Q i$. Since C is finite, there exists at least one cycle $i_1\varphi a_1\varphi i_2\varphi a_2\varphi ... i_k\varphi a_k\varphi i_1$. This implies $i_1 \succ_Q i_k \succ_Q i_{k-1} \succ_Q ... \succ i_2 \succ_Q i_1$, contradiction. $\square$

Proof of Proposition 2. The first statement is obvious. To show the second statement, we first establish the consistency of eating mechanisms. Given $\omega \in \mathcal{W}^{\mathcal{N}}$, let $\bar{f}$ be the extended eating mechanism. For each problem $e = (N, I, R)$, $\bar{f}^w(e)$ is defined the same as $f^w(e)$ in section 4, except $O^0 = \{a \in \mathcal{O} : I_a > 0\}$ and (4.1) is replaced by the following:

(A.1.1) $t_a^k = min \left\{ t : \sum_{i \in M(a, O^{k-1})} \int_{t^{k-1}}^{t} \omega_i(t)dt + \sum_{i \in N'} \pi_{ia}^{k-1} = I_a \right\}$ if $M(a, O^{k-1}) \neq \phi$ and $I_a \neq 0$, $t_a^k = 1$ otherwise.

Given any $e = (N, I, R)$ and $S \subseteq N$, let $(m_i(t))_{i \in N}$ be the consumption schedule representing $\bar{f}^\omega(e)$ at $(\omega_i)_{i \in N}$, and $(\widetilde{m}_i(t))_{i \in S}$ be the consumption schedule representing $\bar{f}^\omega(r_S^{\bar{f}^\omega(e)}(e))$ at $(\omega_i)_{i \in S}$, as defined in (4.5). Then $\pi(t) = (\pi_i(t))_{i \in S}$ and $\widetilde{\pi}(t) = (\widetilde{\pi}_i(t))_{i \in S}$ denote the assignments made at each t during the consumption process for e and $r_S^{\bar{f}^\omega(e)}(e)$, respectively.

Let $\{t^k\}$ be the sequence defined for $r_S^{\bar{f}^\omega(e)}(e)$ as in (4.2). It is sufficient to show that $\pi(t^k) = \widetilde{\pi}(t^k)$ for each k. Obviously $\pi(t^0) = \widetilde{\pi}(t^0)$. Suppose for some $k \geqslant 0$, $\pi(t^k) = \widetilde{\pi}(t^k)$. We want to show that $\pi(t^{k+1}) = \widetilde{\pi}(t^{k+1})$. The consumption schedules $m_i(t)$ and $\widetilde{m}_i(t)$ might not coincide for each $i \in S$, but we have the following result:

Claim 1. *For any $i \in S$, let $\alpha(i) = inf \left\{ t : \int_{t^k}^{t} \omega_i(t)dt > 0 \right\}$ if $\int_{t^k}^{t^{k+1}} \omega_i(t)dt > 0$, $\alpha(i) = t^{k+1}$ otherwise. Suppose for some $\hat{t} \in [t^k, t^{k+1})$, $\pi(t) = \widetilde{\pi}(t)$ for all $t \in [t^k, \hat{t}]$, then $m_i(\hat{t}) = \widetilde{m}_i(\hat{t})$ if $\alpha(i) \leqslant \hat{t}$.*

Proof of claim 1. Given such $\hat{t}$, consider any $i \in S$ with $\alpha(i) \leqslant \hat{t}$. Suppose $m_i(\alpha(i)) = b$. Since $\alpha(i) < 1$, by the definition of the function m_i, there exists $\epsilon > 0$ such that $m_i(\alpha(i) + \epsilon) = m_i(\alpha(i)) = b$. By the definition of $\alpha(i)$, we have $\int_{\alpha(i)}^{\alpha(i)+\epsilon} \omega_i(t)dt > 0$. Since $\pi(\alpha(i)) = \widetilde{\pi}(\alpha(i))$, at least $(\int_{\alpha(i)}^{\alpha(i)+\epsilon} \omega_i(t)dt)$ units of b is available at $\alpha(i)$ for $r_S^{\bar{f}^\omega(e)}(e)$. Since any object that is not exhausted at $\alpha(i)$ for $r_S^{\bar{f}^\omega(e)}(e)$ is not exhausted at $\alpha(i)$ for e either, it follows that $\widetilde{m}_i(\alpha(i)) = b$.

Since by definition no object becomes exhausted during (t^k, t^{k+1}) for $r_S^{\bar{f}^\omega(e)}(e)$ and $\alpha(i) \leqslant \hat{t} < t^{k+1}$, we have $\widetilde{m}_i(\hat{t}) = \widetilde{m}_i(\alpha(i)) = b$. Since $\pi(\hat{t}) = \widetilde{\pi}(\hat{t})$, b is not exhausted at $\hat{t}$

for e either. Thus, $m_i(\hat{t}) = m_i(\alpha(i)) = b$. $\square$

Another induction is needed to complete the proof. Define sequences $\{\tau^s\}_{s=1}^{\hat{s}}$ and $\{A(\tau^s)\}_{s=1}^{\hat{s}}$ as follows. $\tau^1 = min\ \{\alpha(i) : i \in S\}$, $A(\tau^1) = \{i \in S : \alpha(i) = \tau^1\}$; for $s \geqslant 2$, $\tau^s = min\ \{\alpha(i) : i \in S \setminus \cup_{p=1}^{s-1} A(\tau^p)\}$, $A(\tau^s) = \{i \in S : \alpha(i) = \tau^s\}$. Let $\tau^{\hat{s}} = t^{k+1}$ if there does not exist $i \in S$ such that $\alpha(i) = t^{k+1}$. Obviously $\pi(t) = \widetilde{\pi}(t), \forall t \in [t^k, \tau^1]$. Suppose for some $s \geqslant 2$, $\pi(t) = \widetilde{\pi}(t), \forall t \in [t^k, \tau^s]$. Consider the set $T = \{t \in [\tau^s, \tau^{s+1}) : \exists\ i \in \cup_{p=1}^{s} A(\tau^p), m_i(t) \neq \widetilde{m}_i(t)\}$. If $T \neq \phi$, let $\underline{T} = min\ T$, then $\pi(t) = \widetilde{\pi}(t), \forall t \in [t^k, \underline{T}]$. However, by claim 1, this implies $m_i(\underline{T}) = \widetilde{m}_i(\underline{T})$ for all $i \in \cup_{p=1}^{s} A(\tau^p)$, contradiction. Thus $T = \phi$. It follows that $\pi(t) = \widetilde{\pi}(t), \forall t \in [t^k, \tau^{s+1}]$. By induction, $\pi(t^{k+1}) = \widetilde{\pi}(t^{k+1})$. By induction again, $\pi(t^k) = \widetilde{\pi}(t^k)$ for any k, this completes the proof of consistency.

Now given a sd-efficient and ex-ante stable π of the problem $\mathcal{E} = (N, O, Q, R)$, we want to show that there exists $\omega \in \mathcal{W}(Q)$ such that $f^\omega(\mathcal{E}) = \pi$. By Theorem 1 in Bogomolnaia and Moulin (2001) there exists $\omega \in \mathcal{W}^N$ such that $f^\omega(\mathcal{E}) = \pi$. $\omega \in \mathcal{W}(Q)$ if $q = 1$. Assume $q > 1$. Without loss of generality, suppose for any i, $\omega_i(t) = 0$ for $t \in (\frac{1}{q}, 1]$. Define a sequence $\{\omega^l\}_{l=1}^{q}$ as follows. Let $\omega^1 = \omega$; for $l \in \{2, 3, ..., q\}$, $\omega_i^l(t) = \omega_i^{l-1}(t - \frac{1}{q})$ if $i \in Q_k$ and $k \geqslant l$, $\omega_i^l = \omega_i^{l-1}$ otherwise. Then $\omega^q \in \mathcal{W}(Q)$. It is sufficient to show that if $f^{\omega^l}(\mathcal{E}) = \pi$, then $f^{\omega^l}(\mathcal{E}) = f^{\omega^{l+1}}(\mathcal{E})$ for any $l \in \{1, 2, ..., q\}$.

Now suppose for an arbitrary l, $f^{\omega^l}(\mathcal{E}) = \pi$. Let $Q_{\leqslant l} = \cup_{k \leqslant l} Q_k, Q_{>l} = N \setminus Q_{\leqslant l}$.

Step 1. $f_i^{\omega^l}(\mathcal{E}) = f_i^{\omega^{l+1}}(\mathcal{E})$ *for* $i \in Q_{\leqslant l}$.

Let $m^1(t)$ be the consumption schedule representing $f^{\omega^l}(\mathcal{E})$ at ω^l and $m^2(t)$ be the consumption schedule representing $f^{\omega^{l+1}}(\mathcal{E})$ at ω^{l+1}, as defined in (4.5). $(\pi_i^1(t))_{i \in N}$ and $(\pi_i^2(t))_{i \in N}$ denote the assignments made at each t under ω^l and ω^{l+1}, respectively. Since $\omega_i^l = \omega_i^{l+1}$ for all $i \in Q_{\leqslant l}$, it is sufficient to show that $m_i^1(t) = m_i^2(t)$ for all $i \in Q_{\leqslant l}$ and $t \in \left(0, \frac{l}{q}\right)$. Let $T' = \{t \in \left(0, \frac{l}{q}\right) : \exists i \in Q_{\leqslant l},\ m_i^1(t) \neq m_i^2(t)\}$. Suppose $T' \neq \phi$, then let $\underline{T}' = min\ T'$. Thus $\pi_i^1(\underline{T}') = \pi_i^2(\underline{T}')$ for all $i \in Q_{\leqslant l}$, and $m_{i'}^1(\underline{T}') \neq m_{i'}^2(\underline{T}')$ for some $i' \in Q_{\leqslant l}$. Then by the construction of the consumption schedule function in (4.5), $m_{i'}^1(\underline{T}') \neq m_{i'}^2(\underline{T}')$ implies $\sum_{b \in O} \pi_{i'b}^1(\underline{T}') = \sum_{b \in O} \pi_{i'b}^2(\underline{T}') < 1$. Since an object is not exhausted at $\underline{T}'$ under ω^l only if it is not exhausted at $\underline{T}'$ under ω^{l+1}, we have $m_{i'}^2(\underline{T}') P_{i'} m_{i'}^1(\underline{T}')$ and $\sum_{j \in N} \pi^1_{jm_{i'}^2(\underline{T}')}(\underline{T}') = 1$. So there exist some $j \in Q_{>l}$ and $b \in O$ such that $f^{\omega^l}_{jm_{i'}^2(\underline{T}')}(\mathcal{E}) > 0$, $m_{i'}^2(\underline{T}') P_{i'} b$ and $f_{i'b}^{\omega^l}(\mathcal{E}) > 0$, contradicting to $f^{\omega^l}(\mathcal{E})$ being ex-ante stable. Hence, for any $t \in \left(0, \frac{l}{q}\right)$ and $i \in Q_{\leqslant l}$, $m_i^1(t) = m_i^2(t)$. It follows that $f_i^{\omega^l}(\mathcal{E}) = f_i^{\omega^{l+1}}(\mathcal{E})$ for $i \in Q_{\leqslant l}$.

Step 2. $f_i^{\omega^l}(\mathcal{E}) = f_i^{\omega^{l+1}}(\mathcal{E})$ *for* $i \in Q_{>l}$.

Let $e = (N, I, R)$ with $I_a = 1$ for all $a \in O$. Then $\bar{f}^{\omega^l}(e) = f^{\omega^l}(\mathcal{E})$. Step 1 implies that $f_i^{\omega^{l+1}}(\mathcal{E}) = \bar{f}_i^{\omega^l}(r_{Q_{>l}}^{\bar{f}^{\omega^l}(e)}(e))$ for $i \in Q_{>l}$. Consistency of eating mechanisms implies $\bar{f}^{\omega^l}(e)|_{Q_{>l}} = \bar{f}^{\omega^l}(r_{Q_{>l}}^{\bar{f}^{\omega^l}(e)}(e))$, thus $f_i^{\omega^l}(\mathcal{E}) = f_i^{\omega^{l+1}}(\mathcal{E})$ for $i \in Q_{>l}$.

Hence $f^{\omega^l}(\mathcal{E}) = f^{\omega^{l+1}}(\mathcal{E})$. By induction $f^{\omega^q}(\mathcal{E}) = f^{\omega^1}(\mathcal{E}) = \pi$. □

Proof of Proposition 3. The first statement is obvious. For the second statement, it is sufficient to show that for any stable and efficient deterministic assignment π, there exists $\theta \in \Theta(Q)$ such that $f^\theta(\mathcal{E}) = \pi$. Since serial dictatorships characterize the set of efficient assignments (Svensson, 1994), given a stable and efficient π, there exists some ordering θ such that $f^\theta(\mathcal{E}) = \pi$. Construct another ordering θ' such that for any $i, j \in N$, $\theta'(i) < \theta'(j)$ if $i \succ_Q j$, or, $i \sim_Q j$ and $\theta(i) < \theta(j)$. By construction $\theta' \in \Theta(Q)$. Suppose $f^{\theta'}(\mathcal{E}) = \pi' \neq \pi$, then by the efficiency of π' there exists $i \in N$ such that $\pi'(i) P_i \pi(i)$. Let $\pi'(i) = \pi(j)$, then $\theta(j) < \theta(i)$, and by the stability of π, $j \succeq_Q i$. Thus by the construction of θ', $\theta'(j) < \theta'(i)$. Then $\pi'(i) = \pi(j)$ implies $\pi'(j) P_j \pi(j)$. Since N is finite, continue in this fashion there exists a sequence $\{i_1, i_2, ..., i_k\}$ such that $\pi(i_{s+1}) P_{i_s} \pi(i_s)$, $s \in \{1, 2, ..., k-1\}$, and $\pi(i_1) P_{i_k} \pi(i_k)$, contradicting to π being efficient. Hence $f^{\theta'}(\mathcal{E}) = f^\theta(\mathcal{E}) = \pi$, where $\theta' \in \Theta(Q)$. □

Proof of Proposition 4. To show that f^{GRSD} is Bayesian consistent at the top, first consider the following decomposition of f^{GRSD}: for any $\mathcal{E} = (N, O, Q, R)$ and $\pi \in \mathcal{A}(\mathcal{E})$,

$$\widetilde{f}_\pi(\mathcal{E}) = \frac{\left|\{\theta \in \Theta(Q) : f^\theta(\mathcal{E}) = \pi\}\right|}{|\Theta(Q)|}$$

Given any $\mathcal{E} = \left(N, O, Q = \{Q_k\}_{k=1}^q, R\right)$, $S = \cup_{k=1}^n Q_k$ for some $1 \leqslant n < q$, $\pi' \in \mathcal{A}(S, O, Q, R)$ such that there exists some $\pi'' \in \mathcal{A}(\mathcal{E})$ with $\widetilde{f}_{\pi''}(\mathcal{E}) > 0$ and $\pi''|_S = \pi'$, and $\pi \in \mathcal{A}(\mathcal{E})$ with $\pi|_S = \pi'$, let $\mathcal{E}' = (S, O, Q|_S, R|_S)$ and $\mathcal{E}'' = (N \setminus S, O \setminus \pi'(S), Q|_{N\setminus S}, R|_{N\setminus S})$. The following two results are immediate from the definition of f^{GRSD}.

(A.1.2)

$$\left|\{\theta \in \Theta(Q) : f^\theta(\mathcal{E}) = \pi\}\right| = \left|\{\theta \in \Theta(Q|_S) : f^\theta(\mathcal{E}') = \pi'\}\right| \cdot \left|\{\theta \in \Theta(Q|_{N\setminus S}) : f^\theta(\mathcal{E}'') = \pi|_{N\setminus S}\}\right|$$

(A.1.3) $\left|\{\theta \in \Theta(Q) : f^\theta(\mathcal{E})|_S = \pi'\}\right| = \left|\{\theta \in \Theta(Q|_S) : f^\theta(\mathcal{E}') = \pi'\}\right| \cdot \left|\Theta(Q|_{N\setminus S})\right|$

Then by (A.1.2) we have

$$\widetilde{f}_{\pi}(\mathcal{E}) = \frac{\left|\left\{\theta \in \Theta(Q|_S) : f^{\theta}(\mathcal{E}') = \pi'\right\}\right| \cdot \left|\left\{\theta \in \Theta(Q|_{N\setminus S}) : f^{\theta}(\mathcal{E}'') = \pi|_{N\setminus S}\right\}\right|}{|\Theta(Q)|}$$

By (A.1.3) we have

$$\sum_{\bar{\pi}\in\mathcal{A}(\mathcal{E}),\bar{\pi}|_S=\pi'} \widetilde{f}_{\bar{\pi}}(\mathcal{E}) = \frac{\left|\left\{\theta \in \Theta(Q) : f^{\theta}(\mathcal{E})|_S = \pi'\right\}\right|}{|\Theta(Q)|} = \frac{\left|\left\{\theta \in \Theta(Q|_S) : f^{\theta}(\mathcal{E}') = \pi'\right\}\right| \cdot \left|\Theta(Q|_{N\setminus S})\right|}{|\Theta(Q)|}$$

Hence,

$$\frac{\widetilde{f}_{\pi}(\mathcal{E})}{\sum_{\bar{\pi}\in\mathcal{A}(\mathcal{E}),\bar{\pi}|_S=\pi'} \widetilde{f}_{\bar{\pi}}(\mathcal{E})} = \frac{\left|\left\{\theta \in \Theta(Q|_{N\setminus S}) : f^{\theta}(\mathcal{E}'') = \pi|_{N\setminus S}\right\}\right|}{\left|\Theta(Q|_{N\setminus S})\right|} = \widetilde{f}_{\pi|_{N\setminus S}}(\mathcal{E}'')$$

f^{GRSD} is Bayesian consistent at the top.

To see that f^{GPS} is not Bayesian consistent at the top, consider the following simple example. Suppose $N = \{1,2,3,4\}, O = \{a,b,c,d\}, Q_1 = \{1,2\}, Q_2 = \{3,4\}$. Preferences R and the assignment $f^{GPS}(\mathcal{E})$ are given as follows:

R_1 :	a	b	c	d
R_2 :	a	c	b	d
R_3 :	b	c	d	a
R_4 :	c	b	d	a

	a	b	c	d
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0
2	$\frac{1}{2}$	0	$\frac{1}{2}$	0
3	0	$\frac{1}{2}$	0	$\frac{1}{2}$
4	0	0	$\frac{1}{2}$	$\frac{1}{2}$

Let $\widetilde{f}'$ be any decomposition of f^{GPS}. Then $\widetilde{f}'_{\pi^1}(\mathcal{E}) = \widetilde{f}'_{\pi^2}(\mathcal{E}) = \frac{1}{2}$, where $(\pi^1(1), \pi^1(2), \pi^1(3), \pi^1(4)) = (a, c, b, d)$ and $(\pi^2(1), \pi^2(2), \pi^2(3), \pi^2(4)) = (b, a, d, c)$.

Imagine the departure of Q_1 with assignment $\pi^1|_{Q_1}$. If f^{GPS} is Bayesian consistent at the top, then $\widetilde{f}'_{\pi^1|_{Q_2}}(\mathcal{E}' = (Q_2, \{b,d\}, Q_2, R|_{Q_2})) = 1$. However, $f^{GPS}_{3b}(\mathcal{E}') = f^{GPS}_{4b}(\mathcal{E}') = \frac{1}{2}$, contradiction. □

Proof of Proposition 5. It has already been discussed that f^{GPS} satisfies sd-efficiency, sd-envy-free and ex-ante stability. By a similar argument in Bogomolnaia and Heo (2012), it can be easily seen that f^{GPS} satisfies bounded invariance. Now suppose that a mechanism f satisfies the four axioms. Assume to the contrary, there exists some $\mathcal{E} = (N, O, Q = \{Q_k\}_{k=1}^{q}, R)$ such that $f(\mathcal{E}) \neq f^{GPS}(\mathcal{E})$. Let $\omega \in \mathcal{W}(Q)$ and $\omega_i(t) = q$ for all $i \in N$ and t

such that $\omega_i(t) > 0$. Fix N, O and Q, for any $R' \in \mathcal{R}_O^N$, let $m^{R'}(t)$ be a consumption schedule representing $f^\omega(N, O, Q, R')$ at ω and $\widetilde{m}^{R'}(t)$ be a consumption schedule representing $f(N, O, Q, R')$ at ω. Denote $t_{R'} = min \cup_{k=1}^{q} \left\{ t \in \left[\frac{k-1}{q}, \frac{k}{q}\right) : \exists i \in Q_k, m_i^{R'}(t) \neq \widetilde{m}_i^{R'}(t) \right\}$. Let $R^* \in \mathcal{R}_O^N$ be such that $t_{R^*} = min_{R' \in \mathcal{R}_O^N} t_{R'}$. Since $f(\mathcal{E}) \neq f^{GPS}(\mathcal{E})$, $t_{R^*} < 1$. Thus there exists $i \in Q_k$ for some $1 \leqslant k \leqslant q$ such that $m_i^{R^*}(t_{R^*}) \neq \widetilde{m}_i^{R^*}(t_{R^*})$. By claim 1 in remark 1, this implies that $\widetilde{m}_i^{R^*}(t_{R^*})$ is not agent i's best available object at t_{R^*}. Suppose that a is agent i's best available object at t, then $aP_i\widetilde{m}_i^{R^*}(t)$ for all $t \geqslant t_{R^*}$, and there exists some $j \in N$ who consumes a positive share of a after t_{R^*}. So $j \in Q_l$ for some $l \geqslant k$. The argument in Bogomolnaia and Heo (2012) can be applied to show that a contradiction is reached if $l = k$. Suppose $l > k$, then $i \succ_Q j$, $f(\mathcal{E}')_{ja} > 0$, and i consumes a positive share of some b during $\left[t_{R^*}, \frac{k}{q}\right]$ with aP_ib, contradicting to the ex-ante stability of $f(\mathcal{E}')$. $\square$

Proof of Proposition 6. The weak inequality can be established based on a characterization of sd-efficiency by Bogomolnaia and Moulin (2001): given an assignment π of $\mathcal{E} = (N, O, Q, R)$, $\forall a, b \in O$, let $a\gamma(\pi)b$ if for some $i \in N, aP_ib$ and $\pi_{ib} > 0$, then π is sd-efficient if and only if $\gamma(\pi)$ is acyclic.

Fix N and O. Suppose $Q' \in \mathcal{Q}_N$ is a refinement of $Q \in \mathcal{Q}_N$, then $\Theta(Q') \subset \Theta(Q)$, which implies that for any $i \in N, a \in O$ and $R \in \mathcal{R}_O^N$, $f_{ia}^{GRSD}(N, O, Q', R) > 0$ only if $f_{ia}^{GRSD}(N, O, Q, R) > 0$. Therefore, for any $R \in \mathcal{R}_O^N$, if $f^{GRSD}(N, O, Q, R)$ is sd-efficient, then $\gamma(f^{GRSD}(N, O, Q, R))$ is acyclic, and $\gamma(f^{GRSD}(N, O, Q', R))$ is also acyclic, so $f^{GRSD}(N, O, Q', R)$ is sd-efficient. Hence, *eff* $(Q') \geqslant$ *eff* (Q).

Since RSD is sd-efficient for two-agent and three-agent house allocation problems (Bogomolnaia and Moulin, 2001), it can be easily seen that *eff* $(Q) = 1$ if and only if $Q \in \{Q = \{Q_k\}_{k=1}^q \in \mathcal{Q}_N : q \in \{|N|, |N|-1, |N|-2\} \cap \mathbf{R}_{++},$ *there is at most one* k *with* $|Q_k| > 1\}$. Now suppose *eff* $(Q) \neq 1$, a few examples are sufficient to establish the strict inequality. Since Q' is a refinement of Q, there exist some $i, j \in N$ such that $i, j \in Q_k$ for some k, and $i \in Q'_s, j \in Q'_t$ for some $s, t, s < t$.

Case 1: there exists some Q_l *with* $|Q_l| \geqslant 2, l \neq k$. Suppose $l > k$ (the case of $l < k$ can be shown similarly). Let $\{i', j'\} \subseteq Q_l, \{a, b, c, d\} \subseteq O$. Consider the following preference profile $R \in \mathcal{R}_O^N$:

$R_i : c, b,$

$R_j : c, a,$

$R_{i'} : a, b, d,$

$R_{j'} : b, a, d,$

and any other agent $m \in N$ has a distinct $a_m \in O$ as her first choice with $a_m \notin \{a,b,c,d\}$. $f^{GRSD}(N,O,Q,R)$ is not sd-efficient, since $f^{GRSD}_{i'b}(N,O,Q,R) > 0$, $f^{GRSD}_{j'a}(N,O,Q,R) > 0$. However, under Q', $f^{GRSD}_{i'a}(N,O,Q',R) = f^{GRSD}_{j'a}(N,O,Q',R) = 0, \gamma(f^{GRSD}(N,O,Q',R))$ is acyclic, so $f^{GRSD}(N,O,Q',R)$ is sd-efficient. Thus *eff* $(Q') >$ *eff* (Q).

Case 2: $|Q_l| = 1$ *for all* $l \neq k$. Since *eff* $(Q) \neq 1$, we have $|Q_k| \geqslant 4$. Let $\{i',j'\} \subseteq Q_k \setminus \{i,j\}$, $i' \neq j'$ and $\{a,b,c,d\} \subseteq O$. Consider the following preference profile $R \in \mathcal{R}^N_O$:

$R_i : a,b,c,d,$

$R_j : a,b,c,d,$

$R_{i'} : b,a,c,d,$

$R_{j'} : b,a,c,d,$

any other agent $m \in N$ has a distinct $a_m \in O$ as her first choice, $a_m \notin \{a,b,c,d\}$. $f^{GRSD}(N,O,Q,R)$ is not sd-efficient since $f^{GRSD}_{ib}(N,O,Q,R) > 0$, $f^{GRSD}_{i'a}(N,O,Q,R) > 0$. However, as $i \succ_{Q'} j$, if $a\gamma(f^{GRSD}(N,O,Q',R))b$, then $f^{GRSD}_{ia}(N,O,Q',R) = 1$, thus we cannot have $b\gamma(f^{GRSD}(N,O,Q',R))a$. It follows that $f^{GRSD}(N,O,Q',R)$ is sd-efficient, hence *eff* $(Q') >$ *eff* (Q). □

Appendix A.2

The assignments in Example 3:

$f^{GRSD}(N,O,Q,R)$	a	b	c	a'	b'	c'
1	0.25	0.133	0.117	0.25	0.133	0.117
2	0.25	0.133	0.117	0.25	0.133	0.117
3	0.25	0.017	0.233	0.25	0.017	0.233
4	0.25	0.017	0.233	0.25	0.017	0.233
5	0	0.35	0.15	0	0.35	0.15
6	0	0.35	0.15	0	0.35	0.15

$f^{GPS}(N, O, Q, R)$	a	b	c	a'	b'	c'
1	0.25	0.125	0.125	0.25	0.125	0.125
2	0.25	0.125	0.125	0.25	0.125	0.125
3	0.25	0	0.25	0.25	0	0.25
4	0.25	0	0.25	0.25	0	0.25
5	0	0.375	0.125	0	0.375	0.125
6	0	0.375	0.125	0	0.375	0.125

$f^{GRSD}(N, O, Q', R)$	a	b	c	a'	b'	c'
1	0.5	0.167	0.333	0	0	0
2	0	0	0	0.5	0.167	0.333
3	0.5	0	0.5	0	0	0
4	0	0	0	0.5	0	0.5
5	0	0.833	0.167	0	0	0
6	0	0	0	0	0.833	0.167

$f^{GPS}(N, O, Q', R)$	a	b	c	a'	b'	c'
1	0.5	0.25	0.25	0	0	0
2	0	0	0	0.5	0.25	0.25
3	0.5	0	0.5	0	0	0
4	0	0	0	0.5	0	0.5
5	0	0.75	0.25	0	0	0
6	0	0	0	0	0.75	0.25

And we have:

$$\left\|f^{GRSD}(N, O, Q, R) - f^{GPS}(N, O, Q, R)\right\|_1 = 0.008$$
$$\left\|f^{GRSD}(N, O, Q', R) - f^{GPS}(N, O, Q', R)\right\|_1 = 0.083$$
$$\left\|f^{GRSD}(N, O, Q, R) - f^{GPS}(N, O, Q, R)\right\|_2 = 0.4$$
$$\left\|f^{GRSD}(N, O, Q', R) - f^{GPS}(N, O, Q', R)\right\|_2 = 0.664$$
$$\left\|f^{GRSD}(N, O, Q, R) - f^{GPS}(N, O, Q, R)\right\|_3 = 0.088$$

$$\left\| f^{GRSD}(N,O,Q',R) - f^{GPS}(N,O,Q',R) \right\|_3 = 0.235$$

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