Stable and efficient resource allocation under weak priorities*

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Oct. 7, 2017

Abstract

We study the indivisible object allocation problem without monetary transfer, in which each object is endowed with a weak priority ordering over agents. It is well known that stability is generally not compatible with efficiency. We characterize the priority structures for which a stable and efficient assignment always exists, as well as the priority structures that admit a stable, efficient and (group) strategy-proof rule. While house allocation problems and housing markets are two classic families of allocation problems that admit a stable, efficient and group strategy-proof rule, any priority-augmented allocation problem with more than three objects admits such a rule if and only if it is decomposable into a sequence of subproblems, each of which has the structure of a house allocation problem or a housing market.

JEL Classification: C71; C78; D47; D71

Keywords: indivisible object; priority; house allocation; housing market; stability; group strategy-proofness

*I am greatly indebted to Bo Chen and Rajat Deb for their guidance and support. Constructive comments from the advisory editor and two anonymous referees have significantly improved this paper. For helpful discussions and comments, I would also like to thank Lars Ehlers, Eun Jeong Heo, Vikram Manjunath, Debing Ni, Shigehiro Serizawa, Alexander Teytelboym, William Thomson, Utku Ünver, Rodrigo Velez, participants at the seminar at Southern Methodist University, the 12^{th} Meeting of the Society for Social Choice and Welfare and the 2014 Texas Economic Theory Camp. All the remaining errors are mine.

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1 Introduction

Indivisible object allocation problems are often "priority-augmented": while each agent has a preference ordering over a set of heterogeneous and indivisible objects, each object also comes with a priority ordering over the agents. Priorities define the agents' relative claims to the objects and provide guidelines for a normatively appealing allocation rule. An assignment respects the priorities, or is *stable*, if there is no situation in which one agent envies another's assignment for which the first agent has a strictly higher priority. A central difficulty for a mechanism designer in this class of problems is that stability is generally not compatible with efficiency (Roth, 1982, Abdulkadiroğlu and Sönmez, 2003). Therefore, a natural question is that under what conditions a stable and efficient solution exists.[1] When priority orderings are strict, Gale and Shapley (1962)'s *deferred acceptance algorithm* (DA) yields the unique stable assignment that Pareto dominates any other stable assignment. Ergin (2002) shows that DA is efficient (group strategy-proof) if and only if the priority structure is *acyclic*. Thus acyclicity characterizes the priority structures under which a stable and efficient assignment exists for any preference profile, as well as the priority structures that admit a stable, efficient and group strategy-proof rule.

However, agents might have equal claim to some object. Coarse priority rankings are also common in real-world applications: in school choice, a student's priority at a particular school could only be determined by the district and sibling rule; in on-campus dormitory allocation, a group of current residents or senior students might be given equal and higher priority than others. Moreover, two extensively studied families of allocation problems, the *house allocation* problems (Hylland and Zeckhauser, 1979) and the *housing market* problems (Shapley and Scarf, 1974), can both be considered as allocation problems augmented with weak priorities: in house allocation each house ranks all the agents equally; in a housing market each house ranks its owner higher than the other agents. They both admit a stable, efficient and group strategy-proof rule: a *serial dictatorship* for house allocation problems, and *Gale's top trading cycle* for housing markets. Then what are the other priority structures that also admit such an appealing solution? In this study, we consider the allocation problems with weak priorities in the one-to-one matching context and search for solvable priority structures in terms of stability, efficiency and (group) strategy-proofness.[2] We define the *weak non-reversal* condi-

[1]We study restrictions on the priority structures. Heo (2017) considers the maximal preference domain in which stability and efficiency are compatible.

[2]We maintain the usual assumption of strict preferences in the main text. In Section 5.5 we extend the main results to the weak preference domain.

tion and show that it is necessary and sufficient for the existence of a stable and efficient rule, but the set of priority structures that admit a stable, efficient and strategy-proof rule is strictly smaller and characterized by *non-reversal*. Requiring group strategy-proofness further reduces the “maximal domain” of priority structures, which is characterized by *strong non-reversal*.[3]

For the sufficiency parts of the characterizations, we introduce *priority set rules*, which decompose an allocation problem into a sequence of subproblems with simple and familiar structures. Specifically, if a group of agents is ranked higher than all the other agents for every object, it is considered as a *priority set* and we can allocate objects to this set first without violating priorities of the other agents. It turns out that weak non-reversal imposes strong structural requirements on the subproblem induced by the smallest priority set, which can only take one of the three forms: house allocation, housing market and indifference at the top (IT). A general queue allocation procedure, motivated by the *you request my house-I get your turn* (YRMH-IGYT) algorithm (Abdulkadiroğlu and Sönmez, 1999, Sönmez and Ünver, 2005), is introduced to select a stable and efficient assignment for each type of structures, which is reduced to a serial dictatorship for house allocation structures and is equivalent to a top trading cycle mechanism for housing market structures. After the agents in the smallest priority set leave the problem with their assignments, we find the smallest priority set of the reduced problem and repeat this process iteratively.

Priority set rules elicit true preferences for house allocation and housing market structures, but for an IT structure with more than three agents, stable and efficient assignments cannot be selected by any strategy-proof rule. Moreover, when group strategy-proofness is imposed, IT structures are eliminated from any solvable problem if there are at least four objects in the market. Therefore, the two baseline classes of allocation problems, the house allocation problems and the housing market problems, are not merely special structures that admit a stable, efficient and group strategy-proof rule. In general, any priority-augmented allocation problem that admits such a rule must be decomposable into a sequence of subproblems during the iterative process of a priority set rule, each of which has a house allocation or a housing market structure. This idea is further strengthened if weak preferences are allowed. In that case, only certain combinations of house allocation and housing market subproblems admit a stable and efficient rule.

The main implication from our study is that generally we cannot go much beyond allocation problems with private and public endowments. Our results are essentially negative, but it is

[3] In Section 5.3, we further characterize the existence of a stable, efficient and strategy-proof rule that satisfies additional fairness properties: population monotonicity and resource monotonicity.

also worth noting that allowing for weak priorities in the model yields a richer set of solvable problems. When priorities are strict, Ergin's acyclicity is equivalent to the condition that any agent's ranks at two objects can differ at most by one, which implies an almost homogenous priority structure. However, under weak priorities, various combinations of subproblems with different structures yield a larger set of admissible problems, and even strong non-reversal does not impose any bound on how an agent's ranks can differ across objects.

Priority-augmented allocation has been studied extensively in the context of school choice, starting from Abdulkadiroğlu and Sönmez (2003). The incompatibility between stability and efficiency motivates various characterizations of the priority domain. Under strict priorities, there is a complete set of results. In addition to Ergin (2002), the priority structures under which the other two common school choice mechanisms, the top trading cycle mechanism and the Boston mechanism, are stable and efficient are characterized by Kesten (2006) and Kumano (2013), respectively.[4] When agents have multi-unit demands, Kojima (2013) shows that stability is compatible with efficiency or strategy-proofness if and only if the priority structure is essentially homogeneous. However, the literature on weak priorities has been relatively limited. Ehlers and Erdil (2010) also generalize Ergin (2002)'s results to the case of weak priories, but from the perspective that DA is constrained efficient under strict priorities.[5] They characterize the priority structures under which the constrained efficient correspondence is efficient, using an acyclicity condition that is more stringent than strong non-reversal. Ehlers (2006) characterizes the priority structures under which efficiency is compatible with a stronger notion of stability, which requires that an agent does not envy another agent's assignment even if the former agent only has weakly higher priority. Finally, Ehlers and Westkamp (2016) consider the existence of a strategy-proof constrained efficient rule.[6]

Different from these previous studies on the priority domain (except Ehlers and Westkamp (2016)), the questions that we are asking cannot be answered by studying the stability and efficiency of an existing allocation rule.[7] Previously studied rules for weak priorities, such as

[4]Ergin's acyclicity condition is also necessary and sufficient for DA to be immune to various types of manipulation, see Kesten (2012) and Kojima (2011). Moreover, see Haeringer and Klijn (2009) for characterizations of the priority domains in which Nash equilibrium outcomes are stable and efficient under various school choice mechanisms. Hatfield et al. (2016) characterizes the priority structures under which there exists a stable or efficient rule that respects improvements of school quality.

[5]An assignment is constrained efficient if it is stable and not Pareto dominated by any other stable assignment. Under weak priorities there may exist multiple constrained efficient assignments. Erdil and Ergin (2008) introduce a constrained efficient rule: the *stable improvement cycles algorithm*.

[6]They show that under some restrictions, there are at most three types of structures that admit a strategy-proof constrained efficient rule: strict priority structures, housing market structures and IT structures.

[7]Moreover, Ehlers and Westkamp (2016) and the current paper assume one-to-one matching, while the other

DA with a fixed tiebreaking and a constrained efficient solution,[8] are not guaranteed to be both stable and efficient on the weak non-reversal domain. The construction of a priority set rule is central to our results, but it is not entirely new: when strong non-reversal is satisfied, each priority set rule is equivalent to some *hierarchical exchange rule* (Pápai, 2000). Hierarchical exchange rules are the only efficient, group strategy-proof and reallocation-proof rules in the standard one-to-one allocation problem.[9] Thus our results also help identify which exogenous priority structures can be respected by some hierarchical exchange rule.

In the next section, we set up the model and define useful concepts. Section 3 considers the existence of a stable and efficient rule, and Section 4 presents the results when strategy-proofness or group strategy-proofness is imposed. Section 5 provides a further discussion of the results, then Section 6 concludes. All the proofs are contained in Appendix A.

2 Preliminaries

Let $\mathcal{N}$ be a finite set of agents and $\mathcal{H}$ a finite set of objects, or houses. Each house $a \in \mathcal{H}$ has a complete and transitive **priority ordering** $\succeq_a$ on $\mathcal{N}$, with $\succ_a$ and $\sim_a$ denoting its asymmetric and symmetric components, respectively.[10] A **priority structure** $\succeq = (\succeq_a)_{a\in\mathcal{H}}$ is a profile of priority orderings. The **problem** is summarized as $\{\mathcal{N},\mathcal{H},\succeq\}$ and remains fixed for the rest of the paper. Given nonempty $N \subseteq \mathcal{N}$ and nonempty $H \subseteq \mathcal{H}$, let $\succeq_{N,H}$ be the restriction of the priority structure to N and H: $\succeq_{N,H} = (\succeq_a |_N)_{a\in H}$. Then $\left\{N, H, \succeq_{N,H}\right\}$ is a **subproblem** of $\{\mathcal{N},\mathcal{H},\succeq\}$. Denote the set of all the subproblems of $\{\mathcal{N},\mathcal{H},\succeq\}$ as $\mathcal{P}$. Notice that $\{\mathcal{N},\mathcal{H},\succeq\} \in \mathcal{P}$.[11] For any $\left\{N, H, \succeq_{N,H}\right\} \in \mathcal{P}$, each agent $i \in N$ has a complete, transitive and antisymmetric **preference relation** R_i on $H \cup \{i\}$, with P_i denoting its asymmetric component. A house $a \in H$ is **acceptable** to i if $aR_i i$. Let $\mathcal{R}_H^{\{i\}}$ denote the set of all such preference rela-

studies allow multiple copies of each object. This also suggests the difficulty in generalizing our results to the case of many-to-one matching.

[8]DA with a fixed tiebreaking rule is currently used in many school choice programs in the U.S. See Abdulkadiroğlu et al. (2005a) and Abdulkadiroğlu et al. (2005b).

[9]Reallocation-proofness rules out the possibility that two agents can gain by misrepresenting preferences and swapping objects ex post, while one agent will obtain the same assignment if she misrepresents preferences and the other agent reports truthfully.

[10]We abuse the notations slightly when there is no confusion: given $N \subseteq \mathcal{N}, N' \subseteq \mathcal{N}$ and $H \subseteq \mathcal{H}$, denote $N \succ_H N'$ if $i \succ_a j$ for all $i \in N, j \in N'$ and $a \in H$. Similarly we can define $i \succ_a N, N \succeq_H N', i \succeq_H N'$ and so on.

[11]The main focus of this paper is to study the existence of a certain rule for the problem $\{\mathcal{N},\mathcal{H},\succeq\}$, and we solve this problem by considering the existence of such a rule for various types of subproblems with special structures. Therefore, for ease of exposition, our "problem" and "subproblems" do not include preferences, and in the following we define all the concepts for an arbitrary subproblem.

tions. Then $R = (R_i)_{i\in N}$ denotes a **preference profile** and $\mathscr{R}_H^N = \prod_{i\in N} \mathscr{R}_H^{\{i\}}$ is the set of all the preference profiles.

Consider any $\{N, H, \succeq_{N,H}\} \in \mathscr{P}$ and $R \in \mathscr{R}_H^N$. An **assignment** is a one-to-one function $\mu : N \to H \cup N$ such that $\mu(i) \in H \cup \{i\}$ for all $i \in N$. An assignment ν **Pareto dominates** μ if $\nu(i) R_i \mu(i)$ for all $i \in N$ and $\nu(j) P_j \mu(j)$ for some $j \in N$. An assignment is **efficient** if it is not Pareto dominated by any other assignment. μ is **stable** if it satisfies the following conditions: *(i)* **respecting priorities**, $\mu(j) P_i \mu(i)$ implies $j \succeq_{\mu(j)} i$, $\forall i, j \in N$; *(ii)* **individual rationality**, $\mu(i) R_i i$, $\forall i \in N$; *(iii)* **nonwastefulness**, $\mu^{-1}(a) = \phi$ implies $\mu(i) R_i a$, $\forall a \in H, i \in N$.

Given $\{N, H, \succeq_{N,H}\} \in \mathscr{P}$, a **rule** or **mechanism** is a function f that associates an assignment $f(R)$ with each preference profile $R \in \mathscr{R}_H^N$. A rule f is efficient (resp., stable) if for any $R \in \mathscr{R}_H^N$, $f(R)$ is efficient (resp., stable). f is **strategy-proof** if no agent benefits from misrepresenting her preferences, i.e., given any $R \in \mathscr{R}_H^N$, $i \in N$ and $R_i' \in \mathscr{R}_H^{\{i\}}$, $f_i(R) R_i f_i(R_i', R_{-i})$. f is **nonbossy** if no agent can change others' assignments without affecting her own assignment: given any $R \in \mathscr{R}_H^N$, $i \in N$ and $R_i' \in \mathscr{R}_H^{\{i\}}$, $f_i(R) = f_i(R_i', R_{-i})$ implies $f(R) = f(R_i', R_{-i})$. f is **group strategy-proof** if no group of agents can jointly manipulate: given any $R \in \mathscr{R}_H^N$, there do not exist $N' \subseteq N$ and $R_{N'}' \in \mathscr{R}_H^{N'}$ such that $f_i(R_{N'}', R_{-N'}) R_i f_i(R)$ for all $i \in N'$, and $f_j(R_{N'}', R_{-N'}) P_j f_j(R)$ for some $j \in N'$. We also have a weaker form of group strategy-proofness: f is **weakly group strategy-proof** if given any $R \in \mathscr{R}_H^N$, there do not exist $N' \subseteq N$ and $R_{N'}' \in \mathscr{R}_H^{N'}$ such that $f_i(R_{N'}', R_{-N'}) P_i f_i(R)$ for all $i \in N'$.

Lemma 1 (Pápai, 2000) *f is group strategy-proof if and only if f is strategy-proof and nonbossy.*

Barberà et al. (2016) establish a similar result regarding individual and weak group strategy-proofness.[12] Consider the problem $\{\mathscr{N}, \mathscr{H}, \succeq\}$. When $\succeq_a$ is antisymmetric for each $a \in \mathscr{H}$, i.e., the priorities are strict, the following **deferred acceptance algorithm** (DA) of Gale and Shapley (1962) yields the unique stable assignment that Pareto dominates any other stable assignment. Moreover, it is strategy-proof (Dubins and Freedman, 1981, Roth, 1982).

Step 1. Each agent applies to her favorite acceptable house, then each house places the applicant with the highest priority on its waiting list and rejects all the other applicants.

Step k. Each agent who was rejected in the step $k-1$ applies to her next best acceptable house. Each house chooses among the new applicants and the agent already on its waiting list, then places the one with the highest priority on its waiting list and rejects all others.

[12] They show that on a rich preference domain, if a rule satisfies a weaker nonbossy condition and a joint monotonicity condition, then it is strategy-proof if and only if it is weakly group strategy-proof. It follows from this result that the deferred acceptance algorithm described below is weakly group strategy-proof.

The process terminates when every agent (who has at least one acceptable house) is either rejected by all of her acceptable houses or on the waiting list of some house. Then every house is assigned to the agent on its waiting list.

Ergin (2002) characterizes the priority structures under which DA is efficient or group strategy-proof by an acyclicity condition.

Definition 1 An **Ergin-cycle** consists of distinct $i, j, k \in \mathscr{N}$ and distinct $a, b \in \mathscr{H}$ such that $i \succ_a j \succ_a k \succ_b i$.[13] $\succeq$ is **Ergin-acyclic** if there does not exist any Ergin-cycle.

Theorem 1 *(Ergin, 2002) Consider the problem $\{\mathscr{N}, \mathscr{H}, \succeq\}$. Assume $\succeq_a$ is antisymmetric for each $a \in \mathscr{H}$, then the following are equivalent:*
(i) DA is efficient,
(ii) DA is group strategy-proof,
(iii) $\succeq$ is Ergin-acyclic.

Since DA is "agent-optimal stable", it follows directly from Theorem 1 that in the case of strict priorities, Ergin-acyclicity characterizes the priority structures under which a stable and efficient assignment always exists, as well as the priority structures under which a stable, efficient and group strategy-proof rule exists.

3 The existence of a stable and efficient rule

We first consider the following motivating example from Ehlers and Erdil (2010), which illustrates the tension between efficiency and respecting weak priorities.

Example 1 (Ehlers and Erdil, 2010) Suppose there are two houses a, b and three agents i, j, k. The priority structure and preference profile are given as follows.

$\succeq_a$	$\succeq_b$
i, j	k
k	i, j

R_i	R_j	R_k
b	b	a
a	a	b
i	j	k

[13] The scarcity condition in the original definition is omitted here since it is always satisfied in our one-to-one setting.

where i and j are ranked equally for both a and b. It can be easily shown that in this case stability induces efficiency loss. Consider any stable assignment μ. We first observe that if $\mu(k) = a$, then one of i and j's priority for a is violated by k. So $\mu(k) \neq a$. Second, we must have $\mu(k) = b$, because otherwise either b is wasted or k 's priority for b is violated. Finally, a is assigned to either i or j by nonwastefulness. Suppose $\mu(i) = a$. Then there is a Pareto-improving exchange between i and k, but such an exchange is "blocked" by j. Similarly, if $\mu(j) = a$, the assignment is not efficient either.

The priority structure in Example 1 satisfies Ergin-acyclicity, thus a stronger restriction on the priorities is required to ensure that the stability constraints do not induce any welfare loss. In fact, this example is representative: efficiency and stability are compatible if and only if there does not exist such a priority relation.

Definition 2 A **strong priority reversal** consists of distinct $i, j, k \in \mathcal{N}$ and distinct $a, b \in \mathcal{H}$ such that $\{i, j\} \succ_a k \succ_b \{i, j\}$. $\succeq$ satisfies **weak non-reversal** if there does not exist any strong priority reversal.

Note that weak non-reversal implies Ergin-acyclicity. Suppose there is an Ergin-cycle $i \succ_a j \succ_a k \succ_b i$. Then consider j 's priority for b. If $j \succeq_b k$, then $\{j, k\} \succ_b i \succ_a \{j, k\}$. If $k \succ_b j$, then $\{i, j\} \succ_a k \succ_b \{i, j\}$. Therefore, there exists a strong priority reversal.

As given in the proof of Theorem 2, a simple variant of Example 1 implies that weak non-reversal is necessary for the compatibility of efficiency and stability. In the remainder of this section, we focus on constructing a family of stable and efficient rules when the priority structure satisfies weak non-reversal. The basic strategy is to decompose the allocation problem into a sequence of smaller and easier subproblems. When a subset of agents has higher priorities than anyone outside this subset for all the houses, we can allocate houses to this subset first without violating priorities of the other agents. After a stable and efficient assignment is chosen for these agents, they leave the problem, and the process can be repeated for the reduced problem iteratively. We first define such a set of agents, for any subproblem. Given $\left\{N, H, \succeq_{N,H}\right\} \in \mathscr{P}$, for any $i, j \in N$, denote $i \succ\!\succeq_H j$ if $i \succeq_a j$ for all $a \in H$ and $i \succ_b j$ for some $b \in H$.

Definition 3 A nonempty set $S \subseteq N$ is a **priority set** for $\left\{N, H, \succeq_{N,H}\right\} \in \mathscr{P}$ if $i \succ\!\succeq_H j$ for any $i \in S$ and $j \in N \setminus S$.

Lemma 2 *For any $\{N, H, \succeq_{N,H}\} \in \mathscr{P}$, there exists at least one priority set, and if S_1 and S_2 are two priority sets, $S_1 \cap S_2$ is a priority set.*

Since the problem is finite, Lemma 2 implies that, by taking the intersection of all the priority sets, we can find a unique priority set with the smallest number of agents.

Definition 4 A is the **smallest priority set** for $\{N, H, \succeq_{N,H}\} \in \mathscr{P}$ if A is a priority set, and for any priority set S for $\{N, H, \succeq_{N,H}\}$, $A \subseteq S$. $\{N, H, \succeq_{N,H}\} \in \mathscr{P}$ is **minimal** if N is the smallest priority set for $\{N, H, \succeq_{N,H}\}$.

Let $\mathscr{P}^m$ be the set of minimal subproblems of $\{\mathscr{N}, \mathscr{H}, \succeq\}$. If A is the smallest priority set for $\{\mathscr{N}, \mathscr{H}, \succeq\}$, then $\{A, \mathscr{H}, \succeq_{A,\mathscr{H}}\} \in \mathscr{P}^m$, and we can focus on allocation in this subproblem first. Weak non-reversal imposes strong restrictions on the structure of a minimal subproblem, which can only take one of three possible forms: each house ranks all the agents equally, every agent has some house that only ranks her at the top, and every agent has some house that only ranks her at the bottom. For any $\{N, H, \succeq_{N,H}\} \in \mathscr{P}^m$, define $I_{N,H} = \{a \in H : i \sim_a j, \ \forall i, j \in N\}$.

Lemma 3 *$\succeq$ satisfies weak non-reversal if and only if any $\{N, H, \succeq_{N,H}\} \in \mathscr{P}^m$ satisfies one of the following three types of structures*:

(i) (House allocation) $H = I_{N,H}$;

(ii) (Housing market) $|N| \geqslant 2$. For each $i \in N$, there exists nonempty $\mathscr{U}(i) \subseteq H$ such that for any $a \in \mathscr{U}(i)$, $i \succ_a N \setminus \{i\}$, and $j \sim_a k$ for all $j, k \in N \setminus \{i\}$. Moreover, $H = I_{N,H} \cup \{\cup_{i \in N} \mathscr{U}(i)\}$;

(iii) (Indifference at the top, or IT) $|N| \geqslant 3$. For each $i \in N$, there exists nonempty $\mathscr{D}(i) \subseteq H$ such that for any $a \in \mathscr{D}(i)$, $N \setminus \{i\} \succ_a i$, and $j \sim_a k$ for all $j, k \in N \setminus \{i\}$. Moreover, $H = I_{N,H} \cup \{\cup_{i \in N} \mathscr{D}(i)\}$.[14]

Denote the set of minimal subproblems satisfying these three types of structures as $\mathscr{P}^{HA}$, $\mathscr{P}^{HM}$ and $\mathscr{P}^{IT}$, respectively. We first introduce a general queue allocation procedure that generalizes the *you request my house-I get your turn* (YRMH-IGYT) algorithm, which was proposed as a solution for the problem of *house allocation with existing tenants* (Abdulkadiroğlu and Sönmez, 1999, Sönmez and Ünver, 2005), then discuss the mechanics and interpretations of such a procedure for each type of structures. Given $\{N, H, \succeq_{N,H}\} \in \mathscr{P}^{HA} \cup \mathscr{P}^{HM} \cup \mathscr{P}^{IT}$ and an ordering σ

[14]The cardinality conditions in (ii) and (iii) guarantee that a minimal subproblem can satisfy at most one type of structures. Without these conditions, any minimal subproblem with one agent satisfies all the three types, and a minimal subproblem with two agents could satisfy both (ii) and (iii).

of the agents ($\sigma : \{1, 2, ..., |N|\} \to N$, σ is a bijection), a **priority-based queue rule**, or simply a **queue rule**, is defined as follows.

Given a preference profile $R \in \mathscr{R}_H^N$, let agents be assigned their favorite available houses sequentially according to the queue $Q = (\sigma(1), \sigma(2), ..., \sigma(|N|))$ until some agent $\sigma(k)$ demands some house a, for which she is ranked lower than at least one unassigned agent. At this point, the queue $Q' = (\sigma(k), \sigma(k+1), ..., \sigma(|N|))$ is updated by moving those agents in this queue who are ranked higher than $\sigma(k)$ for a to the top of Q', and those moved up agents keep their relative positions the same as in Q'. Then the agents are assigned sequentially according to the updated queue. Generally, whenever an agent demands a house for which she is ranked lower than some unassigned agent, the queue is updated accordingly. If at some point, there is a *loop of queues* $(Q_1, Q_2, ..., Q_n, Q_1)$, i.e., after some n rounds of updating, a queue (Q_1) is updated back to its original form, then let all the agents contributing to the loop (the agents who are moved up at least once in this loop of queues) be assigned their favorite available houses, and proceed with the reduced queue from Q_1.

Denote the resulting assignment from this procedure as $q(\sigma, R)$. First, a minimal subproblem with a house allocation structure is exactly a *house allocation* problem (Hylland and Zeckhauser, 1979). All the agents are ranked equally by each house, and priorities cannot be violated. In this case, a queue rule is reduced to a *serial dictatorship*, which is obviously stable. Moreover, any serial dictatorship is efficient and group strategy-proof (Svensson, 1994, Svensson, 1999). Before discussing the other two types of structures, we provide a simple example for each of them.

Example 2 $N = \{i, j, k, l\} \subseteq \mathscr{N}, H = \{a, b, c, d, e, f, g, h\} \subseteq \mathscr{H}$.

$\{N, H, \succeq_{N,H}\}$ with a housing market structure:

$\succeq_a\|_N$	$\succeq_b\|_N$	$\succeq_c\|_N$	$\succeq_d\|_N$	$\succeq_e\|_N$	$\succeq_f\|_N$	$\succeq_g\|_N$	$\succeq_h\|_N$
i	j	k	l	i,j,k,l	i,j,k,l	i	l
j,k,l	i,k,l	i,j,l	i,j,k			j,k,l	i,j,k

where $\mathscr{U}(i) = \{a, g\}$, $\mathscr{U}(j) = \{b\}$, $\mathscr{U}(k) = \{c\}$, $\mathscr{U}(l) = \{d, h\}$, $I_{N,H} = \{e, f\}$.

$\{N, H, \succeq_{N,H}\}$ with an IT structure:

$\succeq_a\|_N$	$\succeq_b\|_N$	$\succeq_c\|_N$	$\succeq_d\|_N$	$\succeq_e\|_N$	$\succeq_f\|_N$	$\succeq_g\|_N$	$\succeq_h\|_N$
i,j,k,l	j,k,l	i,k,l	i,j,l	i,j,k	i,j,l	i,j,l	i,j,k,l
	i	j	k	l	k	k	

where $\mathscr{D}(i)=\{b\}, \mathscr{D}(j)=\{c\}, \mathscr{D}(k)=\{d,f,g\}, \mathscr{D}(l)=\{e\}$, $I_{N,H}=\{a,h\}$.

A housing market structure has features of both a *housing market* problem (Shapley and Scarf, 1974) and a problem of house allocation with existing tenants, since each agent i can be considered to have an initial endowment set $\mathscr{U}(i)$, but there could also be a set of vacant houses $I_{N,H}$.[15] When there are equal numbers of agents and houses, a housing market structure is exactly a housing market problem. For housing market structures, a queue rule is reduced to the YRMH-IGYT algorithm: a queue is updated only if some agent requests a house owned by another unassigned agent, and the latter agent will be moved to the top of the queue, getting the turn of the former agent. Moreover, each loop of queues $(Q_1, Q_2, ..., Q_n, Q_1)$ generates a trading cycle: the agent at the top of each queue is assigned the house owned by the agent at the top of the next queue in the loop. In fact, given $\{N, H, \succeq_{N,H}\} \in \mathscr{P}^{HM}$, a queue rule can be interpreted as the following **top trading cycle mechanism (TTC)**.

Step 1. Given $R \in \mathscr{R}_H^N$ and an ordering σ, denote $t_1 = \sigma(1)$. Let agent i's initial endowment be $E_i^1 = \mathscr{U}(i)$ if $i \neq t_1$, and $E_{t_1}^1 = \mathscr{U}(t_1) \cup I_{N,H}$. Let agents exchange houses with respect to E^1. Specifically, each agent points at the owner of her favorite house (an agent points at herself if all the houses are not acceptable). There exists at least one cycle since N is finite. Let the set of agents in some cycle be A_1. Then each agent i in A_1 is assigned her favorite house (or herself), $\mu(i)$, and leaves the problem with her assignment.

Step k. Let t_k be the agent with the highest order among $N \setminus \cup_{l=1}^{k-1} A_l$ (according to σ). $E_{t_k}^k = E_{t_k}^{k-1} \cup \left\{\cup_{i \in A_{k-1}} \left\{E_i^{k-1} \setminus \{\mu(i)\}\right\}\right\}$, i.e., t_k inherits all the unassigned endowments of those agents in A_{k-1}; $E_i^k = E_i^{k-1}$ for $i \in N \setminus \cup_{l=1}^{k-1} A_l$ and $i \neq t_k$. Then the remaining agents exchange houses with respect to E^k. Each agent points at the owner of her best remaining house (or herself, if all the remaining houses are not acceptable). Let the set of agents in some cycle be A_k. Each i in A_k is assigned her best remaining house (or herself), $\mu(i)$, and leaves the problem.

The process terminates when all the agents are assigned, then $f^{TTC}(\sigma, R) = \mu$.

Results from Abdulkadiroğlu and Sönmez (1999) can be applied to show that, given any σ and $R \in \mathscr{R}_H^N$, $q(\sigma, R) = f^{TTC}(\sigma, R)$. TTC belongs to the family of *hierarchical exchange rules* from Pápai (2000), which are efficient and group strategy-proof.[16] It is also stable since each agent i is guaranteed to obtain a house weakly better than any house in her initial endowment set $\mathscr{U}(i)$.

[15] The only technical difference from the standard house allocation with existing tenants is that, in a housing market structure, an agent could be the existing tenant of multiple houses.

[16] See Section 5.3 for a detailed discussion of hierarchical exchange rules.

Finally, we consider IT structures. In this context, due to the feature of "indifference at the top", if a queue is updated, then it must be the case that some agent i demands a house in $\mathscr{D}(i)$, and all the other unassigned agents are moved to the top of the queue, or equivalently, agent i is moved to the bottom of the queue. A loop forms at some point of the queue allocation only if each remaining agent demands a house in her $\mathscr{D}$ set, thus there are no conflicting interests and all the remaining agents can be assigned their favorite available houses simultaneously. Specifically, the canonical form of a loop in an IT structure is as follows. Suppose at some point the queue is given by $(x_1, x_2, ..., x_k)$ and x_1 demands a house in $\mathscr{D}(x_1)$, so the queue is updated to $(x_2, ..., x_k, x_1)$. Then x_2 demands a house in $\mathscr{D}(x_2)$, thus the queue is further updated to $(x_3, ..., x_k, x_1, x_2)$. After k rounds of such updating, we are back to the original queue $(x_1, x_2, ..., x_k)$. Queue rules characterize the set of stable and efficient assignments for an IT structure subproblem, and such a result will be convenient for the proof of impossibility results concerning (group) strategy-proofness in the next section.

Lemma 4 *Given $\left\{N, H, \succeq_{N,H}\right\} \in \mathscr{P}^{IT}$, $q(\sigma, \cdot)$ is stable and efficient for any σ. Moreover, for any $R \in \mathscr{R}_H^N$, if μ is stable and efficient, there exists σ such that $q(\sigma, R) = \mu$.*

So far it has been shown that queue rules are stable and efficient for any minimal subproblem when weak non-reversal is satisfied. Given the problem $\{\mathscr{N}, \mathscr{H}, \succeq\}$, where $\succeq$ satisfies weak non-reversal, an ordering σ of the agents and a preference profile $R \in \mathscr{R}_{\mathscr{H}}^{\mathscr{N}}$, we define the **priority set rule** $f^{\succeq}$ by iteratively implementing queue rules.

Step 1. Find the smallest priority set N_1 for $\{\mathscr{N}, \mathscr{H}, \succeq\}$. Apply the queue rule to the minimal subproblem $\left\{N_1, H_1 = \mathscr{H}, \succeq_{N_1,H_1}\right\}$.[17] The resulting assignment is $\mu_1 : N_1 \to H_1 \cup N_1$.

Step *k*. Given the reduced problem $\left\{\mathscr{N} \setminus \cup_{l=1}^{k-1} N_l, H_k = \mathscr{H} \setminus \cup_{l=1}^{k-1} \mu_l(N_l), \succeq_{\mathscr{N} \setminus \cup_{l=1}^{k-1} N_l, H_k}\right\}$, find the smallest priority set N_k. Apply the queue rule to the minimal subproblem $\left\{N_k, H_k, \succeq_{N_k,H_k}\right\}$ and the resulting assignment is $\mu_k : N_k \to H_k \cup N_k$.

The process terminates when every agent is assigned, which takes at most $|\mathscr{N}|$ steps. Then, $f_i^{\succeq}(\sigma, R) = \mu_k(i)$ if $i \in N_k$.

The iterative procedure of finding the smallest priority set and applying the stable and efficient queue rule preserves stability and efficiency for the whole allocation problem.

Proposition 1 *Suppose $\succeq$ satisfies weak non-reversal.* $f^{\succeq}(\sigma, \cdot)$ is stable and efficient for any σ.

[17] In each step, the queue rule is implemented with respect to the restrictions of σ and R to the subproblem.

Therefore, we have finished the sufficiency part of the first characterization result.

Theorem 2 *Consider the problem $\{\mathcal{N}, \mathcal{H}, \succeq\}$. There exists a stable and efficient rule if and only if $\succeq$ satisfies weak non-reversal.*

4 Incentive compatibility

While both serial dictatorships and TTC are group strategy-proof, it can be readily seen that agents may be able to manipulate a queue rule in an IT structure, thus priority set rules are generally not strategy-proof. Consider an IT structure with three agents $\{1,2,3\}$ and three houses $\{a,b,c\}$, where $a \in \mathcal{D}(1)$, $b \in \mathcal{D}(2)$ and $c \in \mathcal{D}(3)$. The preferences are given by aR_1bR_11, bR_2aR_22, aR_33. If $\sigma(i)=i$, then $q_2(\sigma,R)=2$. However, agent 2 can get house a by asserting that it is her first choice.

Unfortunately, as shown in the proof of Theorem 3, for any IT structure with more than three agents, although a stable and efficient assignment always exists, there is no strategy-proof rule to select such an assignment. However, for the case of three agents,[18] DA with a preference-based tiebreaking rule from Ehlers (2006) is stable, efficient and weakly group strategy-proof. Thus, priority set rules can be modified to include such a DA algorithm as the solution to IT structures with three agents instead of queue rules, and such modified priority set rules are weakly group strategy-proof on a smaller priority domain than the weak non-reversal one.

Definition 5 A **priority reversal** consists of some $i,j,k,l \in \mathcal{N}$ and some $a,b,c,d \in \mathcal{H}$ such that $\{i,j,k\} \succ_a l$, $l \succ_b i, l \succ_c j$ and $l \succ_d k$, with $|\{i,j,k\}| \geq 2$, and $|\{b,c,d\}| = 1$ if $|\{i,j,k\}| = 2$. $\succeq$ satisfies **non-reversal** if there does not exist any priority reversal.

Given a priority reversal $\{i,j,k\} \succ_a l$, $l \succ_b i, l \succ_c j$ and $l \succ_d k$, it has a strong priority reversal if $|\{i,j,k\}| = 2$, or, $|\{i,j,k\}| = 3$ and $|\{b,c,d\}| < 3$. Hence non-reversal implies weak non-reversal. It also rules out IT structures with more than three agents.

Theorem 3 *Consider the problem $\{\mathcal{N}, \mathcal{H}, \succeq\}$. The following are equivalent:*
(i) there exists a stable, efficient and strategy-proof rule,
(ii) there exists a stable, efficient and weakly group strategy-proof rule,[19]

[18] Recall that an IT structure consists of at least three agents and three houses.
[19] I thank Lars Ehlers for pointing out that *(ii)* is equivalent to *(i)*.

(iii) $\mathscr{P}^m = \mathscr{P}^{HA} \cup \mathscr{P}^{HM} \cup \left\{\left\{N, H, \succeq_{N,H}\right\} \in \mathscr{P}^{IT} : |N| = 3\right\}$,
(iv) $\succeq$ *satisfies non-reversal.*

Finally, we consider the stronger notion of group strategy-proofness. For the special three-agent and three-house IT case, DA with the preference-based tiebreaking is also group strategy-proof (Ehlers, 2006).[20] It follows that when $|\mathscr{H}| < 4$, weak non-reversal (which is equivalent to non-reversal in this case) is necessary and sufficient for the existence of a stable, efficient and group strategy-proof rule. For a three-agent IT structure with more than three houses, DA with the preference-based tiebreaking is no longer group strategy-proof. In fact, as shown in the proof of Theorem 4, for any IT structure with more than three houses, there does not exist a stable, efficient and group strategy-proof rule, suggesting that the "maximal domain" of priority structures is shrinking further. If some $\left\{N, H, \succeq_{N,H}\right\} \in \mathscr{P}^{IT} \neq \phi$, then $\left\{N, \mathscr{H}, \succeq_{N,\mathscr{H}}\right\}$ is also minimal, and weak non-reversal implies $\left\{N, \mathscr{H}, \succeq_{N,\mathscr{H}}\right\} \in \mathscr{P}^{IT}$. Hence, if $|\mathscr{H}| > 3$, IT structures are eliminated from any solvable problem when group strategy-proofness is required.

Definition 6 A **weak priority reversal** consists of distinct $i, j, k \in \mathscr{N}$ and some $a, b, c \in \mathscr{H}$ such that $\{i, j\} \succ_a k$, $k \succ_b i$ and $k \succ_c j$. $\succeq$ satisfies **strong non-reversal** if there does not exist any weak priority reversal.

Given a weak priority reversal $\{i, j\} \succ_a k, k \succ_b i, k \succ_c j$, it is reduced to a strong priority reversal if $b = c$. Strong non-reversal also rules out all the IT structures and implies that any minimal subproblem is either a house allocation structure or a housing market structure. In light of Lemma 1, the group strategy-proofness of priority set rules for a strong non-reversal problem follows directly from the group strategy-proofness of serial dictatorships and TTC.

Theorem 4 *Consider the problem* $\{\mathscr{N}, \mathscr{H}, \succeq\}$. *Suppose* $|\mathscr{H}| > 3$, *the following are equivalent:*
(i) there exists a stable, efficient and group strategy-proof rule,
(ii) $\mathscr{P}^m = \mathscr{P}^{HA} \cup \mathscr{P}^{HM}$,
(iii) $\succeq$ *satisfies strong non-reversal.*

While house allocation problems and housing markets are two classic families of assignment problems that admit a stable, efficient and group strategy-proof rule, Theorem 4 implies that a partial converse is also true: given any priority-augmented allocation problem (with

[20] It can be shown that, in this case, such DA algorithms are the only stable, efficient and group strategy-proof rules.

more than three houses), such a rule exists only if it can be decomposed as a sequence of subproblems defined by the smallest priority sets, and each subproblem has either a house allocation structure or a housing market structure. A housing market structure is not exactly a housing market problem which features equal numbers of houses and agents. The following corollary reinterprets Theorem 4 and provides a closer connection to these two classic families of problems.

Corollary 1 *Consider the problem* $\{\mathscr{N},\mathscr{H},\succeq\}$. *Suppose* $|\mathscr{H}|>3$, *there exists a stable, efficient and group strategy-proof rule if and only if any* $\left\{N,H,\succeq_{N,H}\right\}\in\mathscr{P}^m$ *with* $|N|=|H|$ *is either a house allocation problem or a housing market problem.*

Strong restrictions on the priority domain are imposed by the full compatibility of stability and efficiency, or stability, efficiency and (group) strategy-proofness, but allowing for weak priorities in the model does yield a much richer set of solvable problems. Define the *rank* of agent i at house a as $r_i(a)=|\{j\in\mathscr{N}:j\succ_a i\}|+1$. Then under strict priorities, Ergin-acyclicity is equivalent to the condition that an agent's ranks at any two houses can differ at most by one (Theorem 2, Ergin (2002)), i.e., $|r_i(a)-r_i(b)|\leq 1$, for any $i\in\mathscr{N}$ and $a,b\in\mathscr{H}$. This condition implies that an Ergin-acyclic (and strict) priority structure is almost homogenous: all the agents are partitioned into several groups $N_1,N_2,...,N_k$, with $N_1\succ_{\mathscr{H}}N_2\succ_{\mathscr{H}}...\succ_{\mathscr{H}}N_k$, and each group has at most two agents. However, under weak priorities, various combinations of minimal subproblems with different structures yield a larger set of admissible problems. Moreover, even strong non-reversal does not impose any bound on the agents' rank differences across houses.[21] The enlarged set of solvable problems is not only because we are considering a larger class of priority structures, but also due to the fact that the stability constraints are less demanding under coarse priorities.

5 Discussion

5.1 More on the compatibility of stability and efficiency

While a stable and efficient rule generally does not exist without restrictions on priorities, weakening the standard stability and efficiency concepts could lead to more positive results.

[21]Rank differences under a strong non-reversal priority structure can be as large as possible. One example is that there are only two houses, house a ranks all the agents equally, and house b ranks all the agents strictly. Then for the agent ranked lowest by b, her rank difference between the two houses is $|\mathscr{N}|-1$.

There is a growing literature on weaker stability notions that are compatible with efficiency.[22] In this subsection we discuss some possible weaker efficiency concepts that are compatible with stability.

First, we can use strong Pareto domination. An assignment μ is **type-I weakly efficient** if there does not exist another assignment ν such that $\nu(i)P_i\mu(i)$ for all $i \in \mathcal{N}$. Type-I weak efficiency is compatible with stability, since applying DA after the ties in priorities are broken in any way will yield a stable and type-I weakly efficient assignment (Roth and Sotomayor, 1990). This also suggests that, if we insist on stable rules, there is a trade-off between notions of efficiency and group strategy-proofness, and the richness of the priority domain. For any priority structure, there exists a type-I weakly efficient and weakly group strategy-proof rule: DA with any fixed tiebreaking. Requiring efficiency and group strategy-proofness restricts the admissible priority structures.

In Example 1, the stable assignment μ, where $\mu(i) = a, \mu(j) = j$ and $\mu(k) = b$, is type-I weakly efficient. Moreover, μ is also "undominated" in the following sense: there is only one possible Pareto-improving exchange, which is an exchange between i and k; however, j is also "involved" in this exchange but she does not benefit from it. This suggests that another weaker efficiency concept can be defined as follows. A stable assignment μ is **type-II weakly efficient** if there does not exist ν such that *(i)* ν Pareto dominates μ, and *(ii)* for any $i, j \in \mathcal{N}$ and $a \in \mathcal{H}$, if $i \succ_a j$, $\nu(j) = a$ and $aP_i\nu(i)$, then $\nu(i)P_i\mu(i)$. Intuitively, a stable assignment is type-II weakly efficient if any possible improvement over the original assignment cannot make all the agents involved in the exchanges (including those whose priorities are violated by the exchanges) strictly better-off.[23] A stable and type-II weakly efficient assignment always exists. In fact, a stable assignment is type-II weakly efficient if and only if it is constrained efficient.[24] While the "only if" part follows from the definitions, the "if" part follows directly from a result regarding dominated stable assignments, which is of some independent interests.

Proposition 2 *Let μ be a stable assignment. If there exists some assignment ν that Pareto dominates μ, and for any $i \in \mathcal{N}$ with $\mu(i) = \nu(i)$, there does not exist $j \in \mathcal{N}, a \in \mathcal{H}$ such that $i \succ_a j, \nu(j) = a$ and $aP_i\nu(i)$, then there exists a stable assignment that Pareto dominates μ.*

[22] See, for example, Kesten (2004), Cantala and Pápai (2014), Alcalde and Romero-Medina (2015), Dur et al. (2015), Morrill (2015), Kloosterman and Troyan (2016), Tang and Zhang (2017), Ehlers and Morrill (2017).

[23] Notice that Pareto-improving a stable assignment is only possible through reshuffling houses among the agents, i.e., exchanges. In fact, it can be easily seen that if an assignment μ is individually rational and nonwasteful, and ν Pareto dominates μ, then $\mu(i) = i$ if and only if $\nu(i) = i$ for all $i \in \mathcal{N}$.

[24] See Section 5.2 for more discussion on constrained efficiency.

Proposition 2 says that, given a stable assignment, if there exist some Pareto-improving exchanges among a subset of agents $N \subseteq \mathscr{N}$ which do not violate the priorities of those agents not in N, then there exist Pareto-improving exchanges which do not violate anyone's priority. In the proof, we strengthen a main result from Erdil and Ergin (2008).[25] They show that if a stable assignment μ is Pareto dominated by another stable assignment, i.e., some subset of agents can exchange their houses at μ such that these exchanges do not violate the priority of any agent in $\mathscr{N}$, then μ admits a stable improvement cycle.[26] In the proof we show that if some subset of agents N can exchange their houses at μ such that the exchanges do not violate the priorities of those agents in $\mathscr{N} \setminus N$, then μ admits a stable improvement cycle. Finally, it follows directly from Proposition 2 that any constrained efficient assignment is type-I weakly efficient. For the case of strict priorities, this result is shown in Roth and Sotomayor (1990).[27] However, a stable and type-I weakly efficient assignment might not be constrained efficient.

5.2 Relationship with other acyclicity conditions

We briefly discuss several acyclicity conditions proposed for the case of strict priorities first.[28] Kesten (2006) shows that TTC is stable if and only if $\succeq$ is *Kesten-acyclic*: there does not exist $i, j, k \in \mathscr{N}$ and $a, b \in \mathscr{H}$ such that $i \succ_a j \succ_a k \succ_b \{i, j\}$. Kumano (2013) shows that the Boston Mechanism is stable or strategy-proof if and only if $\succeq$ is *Kumano-acyclic*: there does not exist distinct $i, j, k \in \mathscr{N}$ and $a, b \in \mathscr{H}$ such that $i \succ_a j \succ_b k$. Let $\mapsto$ denote the "implies but not implied by" relation, then it is easy to see the following: Kumano-acyclicity $\mapsto$ strong non-reversal $\mapsto$ non-reversal $\mapsto$ weak non-reversal $\mapsto$ Ergin-acyclicity $\mapsto$ Kesten-acyclicity.[29]

[25] Results in this subsection can be easily extended to many-to-one matching setting.

[26] Given μ, a stable improvement cycle is a list of agents $i_1, i_2, ..., i_k, i_{k+1} = i_1$ such that for each $l \in \{1, 2, ..., k\}$, $\mu(i_{l+1}) P_{i_l} \mu(i_l)$, and $i_l \succeq_{\mu(i_{l+1})} j$ for all $j \in \mathscr{N}$ with $\mu(i_{l+1}) P_j \mu(j)$.

[27] Our proof of Proposition 2 uses the same technique as theirs. Moreover, under weak priorities, Ehlers (2006) shows that given any constrained efficient assignment, there exists a way to break ties such that DA selects this assignment (but this does not imply DA with some tiebreaking is a constrained efficient rule). Therefore, combining these two results from Roth and Sotomayor (1990) and Ehlers (2006), it also follows that any constrained efficient assignment is type-I weakly efficient.

[28] We consider connections with other restrictions on priority structures. Alcalde and Barberà (1994) consider the two-sided matching problem and identify a *top dominance* condition on the preference domain under which a stable and strategy-proof rule exists. Top dominance restricts the variability of individual preferences such that individual manipulation under DA is avoided, and there is no logical relation between top dominance and non-reversal conditions.

[29] We are comparing these conditions under weak priorities environment. When priorities are strict, strong non-reversal, non-reversal and weak non-reversal are all reduced to Ergin-acyclicity, and Kumano (2013) provides an excellent comparison of various acyclicity conditions under strict priorities.

Ehlers and Erdil (2010) also generalize the results of Ergin (2002) to the case of weak priorities, but from the perspective that DA is the constrained efficient rule under strict priorities. Constrained efficient assignment is generally not unique when ties are allowed, and they show that the constrained efficient correspondence is efficient if and only if the priority structure is *EE-acyclic*: there are no distinct $i, j, k \in \mathscr{N}$ and $a, b \in \mathscr{H}$ such that $i \succeq_a j \succ_a k \succeq_b i$. EE-acyclicity is logically unrelated to Kumano-acyclicity but more stringent than strong non-reversal. Therefore, for some priority structure that fails to satisfy EE-acyclicity, not every constrained efficient assignment is efficient, but there could exist some systematic efficient (and group strategy-proof) selection from the set of constrained efficient assignments.

Finally, Ehlers and Westkamp (2016) consider the same allocation problem and study the priority structures for which there exists a strategy-proof constrained efficient rule. Such a rule exists if a stable, efficient and strategy-proof rule exists, thus there should be a larger set of admissible priority structures than those specified in Theorem 3. One necessary condition provided by them is an acyclicity condition. A tie $i_1 \sim_a i_2$ between two distinct agents i_1, i_2 is *cyclic*, if there exist agents $j_1, j_2 \in \mathscr{N} \setminus \{i_1, i_2\}$, and houses b_1, b_2 such that either $i_1 \succ_{b_1} j_1 \succ_a i_1$ and $i_2 \succ_{b_2} j_2 \succ_a i_2$, with $b_1 = b_2$ if $j_1 = j_2$, or $\{i_1, i_2\} \succ_{b_1} j_1 \succ_{b_2} j_2 \succ_a i_1$. $\succeq$ is *EW-acyclic* if it does not contain a cyclic tie. This new notion helps to establish a characterization of weak non-reversal.

Proposition 3 *$\succeq$ satisfies weak non-reversal if and only if it is Ergin-acyclic and EW-acyclic.*

Therefore, for those priority structures that satisfy Ergin-acyclicity but not weak non-reversal, there does not exist a stable and efficient rule, nor do they admit a strategy-proof constrained efficient rule.

5.3 Priority set rules and hierarchical exchange rules

Hierarchical exchange rules of Pápai (2000) generalize Gale's top trading cycle algorithm by allowing endowment sets to be determined hierarchically by *inheritance trees*. For each house $a \in \mathscr{H}$, an inheritance tree $\Gamma_a = (V, Q)$ is a rooted tree, where V is the set of vertices, and $Q \subset V \times V$ is the set of arcs. Each vertex is labeled by an agent, and each arc is labeled by a house other than a.[30] Γ_a specifies how a is inherited, and the inheritance can endogenously depend on the previous assignments. Given a list of inheritance trees $\Gamma = (\Gamma_a)_{a \in \mathscr{H}}$, the associated

[30] For simplicity we assume all the houses are acceptable to each agent.

hierarchical exchange rule f^{Γ} determines the allocation through top trading cycles at each stage, according to the endowment sets specified by Γ.[31] A rule is efficient, group strategy-proof and reallocation-proof if and only if it is a hierarchical exchange rule.[32]

When the priority structure satisfies strong non-reversal, it can be easily seen that a priority set rule is a hierarchical exchange rule: given the strong non-reversal $\succeq$ and an ordering σ, there exists a corresponding Γ such that $f^{\succeq}(\sigma, \cdot) = f^{\Gamma}$. We briefly discuss the construction of such inheritance trees. Consider any $a \in \mathscr{H}$. Suppose A is the smallest priority set for $\{\mathscr{N}, \mathscr{H}, \succeq\}$. First, a is given to its *top priority agent* in $\left\{A, \mathscr{H}, \succeq_{A,\mathscr{H}}\right\}$ as initial endowment, i.e., a is given to $i \in A$ if $\left\{A, \mathscr{H}, \succeq_{A,\mathscr{H}}\right\} \in \mathscr{P}^{HM}$ and $a \in \mathscr{U}(i)$, or, $a \in I_{A,\mathscr{H}}$ and i has the highest order among A. The inheritance of a is then restricted to the smallest priority set A and follows the ordering σ.[33] It can be easily seen that a hierarchical exchange rule associated with such inheritance trees is equivalent to $f^{\succeq}(\sigma, \cdot)$ for the subproblem $\left\{A, \mathscr{H}, \succeq_{A,\mathscr{H}}\right\}$. The key part of constructing an equivalent hierarchical exchange rule for the whole problem utilizes the fact that the inheritance of a house can depend on the previous agents' assignments. Suppose j is the agent with the lowest order among A. Then given any inheritance path from the top priority agent to j, and any possible assignment of j (other than a), a complete "history" of assignments for A is also known. Therefore, conditional on any possible assignment μ for A, $a \notin \mu(A)$, we can find the smallest priority set A' for the reduced problem $\left\{\mathscr{N} \setminus A, \mathscr{H} \setminus \mu(A), \succeq_{\mathscr{N}\setminus A, \mathscr{H}\setminus\mu(A)}\right\}$, and let a be inherited from j to the top priority agent of a in $\left\{A', \mathscr{H} \setminus \mu(A), \succeq_{A', \mathscr{H}\setminus\mu(A)}\right\}$. The inheritance then follows σ again, within A', and by repeating this procedure an equivalent hierarchical exchange rule is obtained. The following example illustrates this method.

Example 3 $\mathscr{N} = \{i, j, k, l\}$, $\mathscr{H} = \{a, b, c, d\}$. Consider the following strong non-reversal priority structure $\succeq$:

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
i, j, k, l	i	j	i, j, k
	j	i, l	l
	l	k	
	k		

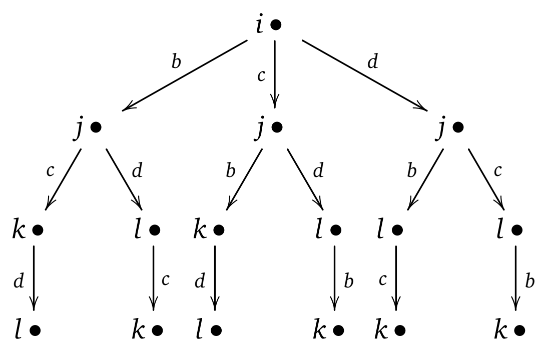
[31] We refer to Pápai (2000) for the formal definition of hierarchical exchange rules.

[32] In a recent work, Pycia and Ünver (2017) introduce a more general class of rules called *trading cycles*, which are the only efficient and group strategy-proof rules.

[33] Specifically, suppose i is the top priority agent of a, then list the set of agents $A \setminus \{i\}$ as $x_1, x_2, ..., x_{|A|-1}$ according to σ. If i is assigned some house other than a, then a is inherited to x_1. Generally, if x_k is assigned some house other than a, then a is inherited to $x_{k+1}, k = 1, 2, ..., |A| - 2$.

Let $(\sigma(1), \sigma(2), \sigma(3), \sigma(4)) = (i, j, k, l)$. Then given σ, there exist inheritance trees Γ corresponding to $\succeq$, such that $f^{\succeq}(\sigma, \cdot) = f^{\Gamma}$. Γ_a is given as follows.

Γ_a



The smallest priority set for $\{\mathscr{N}, \mathscr{H}, \succeq\}$ is $A = \{i, j\}$, and i is the top priority agent of a. The inheritance of a then follows σ within A: a is inherited to j when i is assigned a house other than a. Then the inheritance from j depends on the assignment of A. For instance, if i and j are assigned b and c, respectively, then the smallest priority set for the reduced problem $\left\{N = \{k, l\}, H = \{a, d\}, \succeq_{N,H}\right\}$ is $A' = \{k\}$, hence a is inherited from j to k.

Since any hierarchical exchange rule is efficient and group strategy-proof, it follows that there exists a stable hierarchical exchange rule for the problem $\{\mathscr{N}, \mathscr{H}, \succeq\}$, $|\mathscr{H}| > 3$, if and only if $\succeq$ satisfies strong non-reversal. Although Pápai (2000) does not formally include priorities in her model, hierarchical exchange rules are introduced as a broader class of solutions that are more flexible and less discriminating compared to serial dictatorships, and they can accommodate some exogenous priorities or property rights by specifying proper inheritance trees. Our result shows formally which exogenous priority structures can be respected in the sense of stability. While serial dictatorships and Gale's top trading cycle are two extreme cases of hierarchical exchange rules, a stable hierarchical exchange rule exists only if the problem can be decomposed into a sequence of subproblems such that for each subproblem, one of the two extremes can be applied.

5.4 Population monotonicity and resource monotonicity

Besides stability, another interesting fairness notion for indivisible object allocation problems is *population-monotonicity*. It requires that the agents are affected in the same direction when there is an exogenous change in population. When efficiency is further required, it implies that all the agents are weakly better-off if the population shrinks, and all the agents are weakly worse-off if the population expands. To allow variable population, given $\{\mathscr{N}, \mathscr{H}, \succeq\}$, we interpret $\mathscr{N}$ as the set of *potential* agents, while $\mathscr{H}$ and $\succeq$ remain fixed. For each $N \subseteq \mathscr{N}$ and $R \in \mathscr{R}^N_{\mathscr{H}}$, an (extended) rule $\widetilde{f}$ selects an assignment $\widetilde{f}(N,R)$ for the subproblem $\left\{N, \mathscr{H}, \succeq_{N,\mathscr{H}}\right\}$. $\widetilde{f}$ is **population-monotonic** if for any $M \subseteq N \subseteq \mathscr{N}$ and $R \in \mathscr{R}^N_{\mathscr{H}}$, either $\widetilde{f}_i(M,R|_M)R_i\widetilde{f}_i(N,R)$ for all $i \in M$, or $\widetilde{f}_i(N,R)R_i\widetilde{f}_i(M,R|_M)$ for all $i \in M$.

Ehlers et al. (2002) show that *restricted endowment inheritance rules* are the only efficient, strategy-proof and population-monotonic rules. Each restricted endowment inheritance rule is a hierarchical exchange rule defined with respect to a strict and Ergin-acyclic priority structure. Given such a priority structure $\succeq$, for any $N \subseteq \mathscr{N}$, the inheritance of each house a within N follows the ordering $\succeq_a|_N$. As discussed below Corollary 1, a strict and Ergin-acyclic priority structure implies a partition of agents into singletons and pairs. Hence, under a restricted endowment inheritance rule with respect to some strict and Ergin-acyclic $\succeq$, in any population at most two agents can exchange houses at a time, and the priorities in $\succeq$ are respected.

$\succeq'$ is a **resolution** of $\succeq$ if $\succeq'$ is a strict priority structure, and for any $i, j \in \mathscr{N}$ and $a \in \mathscr{H}$, $i \succ_a j$ implies $i \succ'_a j$. Then a stable, efficient, population-monotonic and strategy-proof rule exists if $\succeq$ has an Ergin-acyclic resolution. The converse is also true: if such a rule exists, then by the characterization result in Ehlers et al. (2002), it must be a restricted endowment inheritance rule with respect to some strict and Ergin-acyclic $\succeq'$; then $\succeq'$ is a resolution of $\succeq$.[34] It can also be easily shown that $\succeq$ admits an Ergin-acyclic resolution if and only if any minimal subproblem is either a house allocation structure, or a housing market structure with two agents.

Similar results hold for *resource-monotonicity*, which requires that the agents are affected in the same direction when there is an exogenous change in resources. Now, we fix $\mathscr{N}$ and $\succeq$, and interpret $\mathscr{H}$ as the set of *potential* houses. For each $H \subseteq \mathscr{H}$ and $R \in \mathscr{R}^{\mathscr{N}}_H$, an (extended) rule $\widetilde{f}$ selects an assignment $\widetilde{f}(H,R)$ for the subproblem $\left\{\mathscr{N}, H, \succeq_{\mathscr{N},H}\right\}$. Then $\widetilde{f}$ is **resource-monotonic** if for any $H' \subseteq H \subseteq \mathscr{H}$ and $R \in \mathscr{R}^{\mathscr{N}}_H$, $\widetilde{f}_i(H',R|_{H'})R_i\widetilde{f}_i(H,R)$ for all $i \in \mathscr{N}$,

[34] If $\succeq'$ is not a resolution of $\succeq$, then there exist some i, j and a such that $i \succ_a j$ but $j \succ'_a i$. Suppose $N = \{i, j\}$ and they both rank a as the best house, then j is assigned a under this restricted endowment inheritance rule, which is not stable.

or $\widetilde{f}_i(H,R) R_i \widetilde{f}_i(H', R|_{H'})$ for all $i \in \mathcal{N}$. Ehlers and Klaus (2004) show that the restricted endowment inheritance rules, adapted to the context of variable resources, are characterized by efficiency and resource-monotonicity.[35] Hence, by similar arguments as before, there exists a stable, efficient and resource-monotonic rule if and only if $\succeq$ has an Ergin-acyclic resolution.

Given the results in Ergin (2002), it is obvious that having an Ergin-acyclic resolution is sufficient for a weak priority structure to admit a stable and efficient rule. But the main results in this paper indicate that it is not a necessary condition. However, if population-monotonicity or resource-monotonicity, as well as strategy-proofness, is further required, then only those structures having an Ergin-acyclic resolution are admissible.

5.5 Weak preferences

While some agents may have equal claim to a house, an agent could also be indifferent among some houses. In this subsection we extend the main results of this paper to the case where the orderings from both sides of the market are weak. Suppose that for each $i \in \mathcal{N}$, each preference relation $R_i \in \mathscr{R}_{\mathscr{H}}^{\{i\}}$ on $\mathscr{H} \cup \{i\}$ is complete and transitive, but not necessarily antisymmetric. Let I_i denote the symmetric part of R_i. The following example shows that a stable and efficient assignment no longer exists in some previously admissible priority structures.

Example 4 Suppose that there are two houses a, b and three agents i, j, k. The priorities and preference profile are given as follows.

$$\begin{array}{llcll} a: & i \succeq_a j \succ_a k & & i: & a I_i b P_i i \\ b: & i \succeq_b k \succ_b j & & j: & b P_j a P_j j \\ & & & k: & a P_k b P_k k \end{array}$$

There are three potentially stable assignments: *(1)* $\mu(i) = i$, $\mu(j) = a$, $\mu(k) = b$; *(2)* $\mu(i) = a$, $\mu(j) = j$, $\mu(k) = b$; *(3)* $\mu(i) = b$, $\mu(j) = a$, $\mu(k) = k$. But none of them is efficient.

In order for a stable and efficient rule to exist, it is necessary to rule out both strong priority-reversal and the priority relations in Example 4. It can be seen that this is equivalent to requiring that there do not exist distinct $i, j, k \in \mathcal{N}$ and $a, b \in \mathscr{H}$ such that $i \succeq_a j \succ_a k \succ_b j$. It turns out that this is also sufficient for the existence of a stable, efficient and strategy-proof rule.

[35] Their characterization also uses *independence of irrelevant houses*, which is a natural requirement that is implicitly assumed in our definition of an extended rule under variable resources.

Theorem 5 *Consider the problem* $\{\mathcal{N},\mathcal{H},\succeq\}$, *and assume each agent can have weak preferences. Then the following are equivalent:*
(i) there exists a stable and efficient rule,
(ii) there exists a stable, efficient and strategy-proof rule,
(iii) there do not exist distinct $i,j,k\in\mathcal{N}$ *and* $a,b\in\mathcal{H}$ *such that* $i\succeq_a j\succ_a k\succ_b j$.

The idea behind the priority set rules may not work under weak preferences, since assigning houses to some agents first without considering the preferences of the other agents cannot guarantee efficiency. However, condition *(iii)* implies a very restrictive priority structure. In particular, if A is the smallest priority set for $\{\mathcal{N},\mathcal{H},\succeq\}$, then $\left\{A,\mathcal{H},\succeq_{A,\mathcal{H}}\right\}$ can only be a house allocation or a housing market structure. Moreover, priorities over the other agents cannot have any reversal. That is, for any $i,j\in\mathcal{N}\setminus A$ and $a,b\in\mathcal{H}$, $i\succ_a j$ implies $i\succeq_b j$. If $\left\{A,\mathcal{H},\succeq_{A,\mathcal{H}}\right\}$ is a house allocation structure, clearly we can find some ordering σ of the agents such that for any $i,j\in\mathcal{N}$ and $a\in\mathcal{H}$, $i\succ_a j$ implies $\sigma^{-1}(i)<\sigma^{-1}(j)$. Then the serial dictatorship rule defined in Svensson (1994) for the weak domain, with respect to σ, is stable, efficient and strategy-proof for $\{\mathcal{N},\mathcal{H},\succeq\}$.

On the other hand, if $\left\{A,\mathcal{H},\succeq_{A,\mathcal{H}}\right\}$ is a housing market structure, then the whole problem $\{\mathcal{N},\mathcal{H},\succeq\}$ can be treated as a complicated allocation problem with both private and public endowments, and a more sophisticated top trading cycle mechanism can be applied. Jaramillo and Manjunath (2012) propose *top cycles rules* for the weak preference domain, which are stable, efficient and strategy-proof for any housing market problem with equal numbers of agents and houses. The top cycles rules are not directly applicable, since in our problem $\{\mathcal{N},\mathcal{H},\succeq\}$, *(1)* some house may not have an initial owner, *(2)* some agent may not own any house, *(3)* an agent may own multiple houses, and *(4)* there are other differences in priorities, besides those coming from initial endowments, that need to be respected. As shown in Jaramillo and Manjunath (2012), *(1)* and *(2)* can be solved by adding dummy agents and dummy houses, respectively. In the proof of Theorem 5, we show that *(3)* can also be handled by adding both dummy agents and dummy houses, and *(4)* can be handled by implementing the top cycles rule with respect to a properly selected ordering of the agents. In sum, $\{\mathcal{N},\mathcal{H},\succeq\}$ can be transformed to a housing market with equal numbers of agents and houses, and applying some top cycles rule to the transformed problem induces a stable, efficient and strategy-proof rule for the original problem.

Theorem 5 sharpens the main implication from Theorem 4. When weak preferences are allowed, only a few combinations of house allocation and housing market subproblems admit a stable and efficient rule. Group strategy-proofness is not required to reach this conclusion.[36] Moreover, the serial dictatorship rules from Svensson (1994) are special cases of top cycles rules. Hence the top cycles rules from Jaramillo and Manjunath (2012) are in fact general enough: whenever a stable and efficient rule exists under weak preferences, some (generalizations of) top cycles rules are stable and efficient. This also corresponds to the result regarding hierarchical exchange rules in the strict preference domain, discussed in Section 5.3, and further confirms the importance of trading procedures.

6 Conclusion

We have considered a generalized house allocation model and searched for solvable problems in terms of stability and efficiency. When group strategy-proofness is further required, solvable problems feature a decomposition into two extensively studied families of allocation problems: one with social endowments and one with private endowments. An interesting and challenging question for future study would be to generalize the results to the case of many-to-one matching. One difficulty imposed by weak priorities is that it is not clear what kind of rule can select a stable and efficient assignment for solvable problems, but in light of our results as well as Pycia and Ünver (2017), it is reasonable to conjecture that such a rule involves some top trading cycle procedure if group strategy-proofness is required. When multiple copies of each object are allowed, and resources are thus less scarce, more priority structures become admissible. While similar characterizations of the priority domain could be established, the set of problems that admit a stable, efficient and group strategy-proof rule will generally not be restricted to combinations of house allocation and housing market problems.

Appendix A

Proof of Lemma 2. Given $\left\{N, H, \succeq_{N,H}\right\} \in \mathscr{P}$, by definition N is a priority set. For any two priority sets S_1 and S_2, we have $S_1 \cap S_2 \neq \phi$, since otherwise there exist some $i \in S_1, j \in S_2$ such that $i \succ_{\succeq_H} j$ and $j \succ_{\succeq_H} i$, contradiction. $S_1 \cap S_2$ is a priority set if $S_1 \cap S_2 = N$. If $S_1 \cap S_2 \neq N$, then for any $k \in S_1 \cap S_2$ and $l \in N \setminus \{S_1 \cap S_2\}$, we have $l \in N \setminus S_1$ or $l \in N \setminus S_2$, so $k \succ_{\succeq_H} l$. Thus $S_1 \cap S_2$ is a priority set. □

[36] When preferences are weak, group strategy-proofness is not compatible with efficiency (Ehlers, 2002).

Proof of Lemma 3. "if" part. Suppose there exists a strong priority reversal, i.e., there exist distinct $i, j, k \in \mathcal{N}$ and distinct $a, b \in \mathcal{H}$ such that $\{i, j\} \succ_a k \succ_b \{i, j\}$. Then the subproblem $\left\{N = \{i, j, k\}, H = \{a, b\}, \succeq_{N,H}\right\}$ is minimal, but it does not satisfy any of the three types of structures.

"only if" part. Suppose $\succeq$ satisfies weak non-reversal and $\left\{N, H, \succeq_{N,H}\right\} \in \mathscr{P}^m$. We first show the following two claims.

Claim 1 (i) *There are no $i, j \in N$ such that $i \succ\succeq_H j$, and (ii) there are no $a \in H$ and $i, j, k \in N$ such that $i \succ_a j \succ_a k$.*

Proof of Claim 1. *Part* (i). Assume to the contrary, there exist agents $i, j \in N$ such that $i \succ\succeq_H j$. Let $G_i = \{k \in N : k \succ\succeq_H j\}, G_j = N \setminus G_i$, then $i \in G_i, j \in G_j$. Given any $l \in G_j$, either $l \sim_H j$ or there exists some $a \in H$ such that $j \succ_a l$. If $l \sim_H j$, then $k \succ\succeq_H l$ for all $k \in G_i$. If there exists some $a \in H$ such that $j \succ_a l$, then there does not exist any $k \in G_i$ and $b \in H \setminus \{a\}$ such that $l \succ_b k$, since otherwise we have $l \succ_b \{k, j\}$ and $\{k, j\} \succ_a l$, which is a strong priority reversal. So $G_i \succeq_H l$ and $G_i \succ_a l$, thus $k \succ\succeq_H l$ for all $k \in G_i$. This shows that $k \succ\succeq_H k'$, for any $k \in G_i$ and $k' \in G_j$. Hence G_i is a strictly smaller priority set than N, contradicting to $\left\{N, H, \succeq_{N,H}\right\}$ being minimal. *Part* (ii) follows immediately: if there exist $a \in H$ and $i, j, k \in N$ such that $i \succ_a j \succ_a k$, then by Ergin-acyclicity (recall that Ergin-acyclicity is implied by weak non-reversal), $i \succ\succeq_H k$, contradicting to part (i). □

For any $a \in H$, if $a \notin I_{N,H}$, then by (ii) of Claim 1 we can partition N into two disjoint nonempty subsets A_a^1, A_a^2, such that $A_a^1 \cup A_a^2 = N$, $A_a^1 \succ_a A_a^2$ and $i \sim_a j$ for any $i, j \in A_a^n, n = 1, 2$.

Claim 2 *For any $a \in H \setminus I_{N,H}$, we cannot have $\left|A_a^1\right| \geqslant 2$ and $\left|A_a^2\right| \geqslant 2$.*

Proof of Claim 2. Assume to the contrary, $\left|A_a^1\right| \geqslant 2$ and $\left|A_a^2\right| \geqslant 2$ for some $a \in H \setminus I_{N,H}$. Then without loss of generality, let $\{i_1, i_2\} \subseteq A_a^1$, $\left\{i_3, i_4\right\} \subseteq A_a^2$. By (i) of Claim 1 we cannot have $i_1 \succ\succeq_H i_3$, so there exists $b \in H$ such that $i_3 \succ_b i_1$, thus $i_3 \in A_b^1, i_1 \in A_b^2$. Then weak non-reversal implies $i_2 \in A_b^1, i_4 \in A_b^2$. Since $i_2 \succ_{\{a,b\}} i_4$, by (i) of Claim 1, there exists $c \in H$ such that $i_4 \succ_c i_2$, so $i_4 \in A_c^1, i_2 \in A_c^2$. Then since $\{i_1, i_2\} \succ_a i_4$, weak non-reversal implies $i_1 \in A_c^1$. Hence, we have $\left\{i_1, i_4\right\} \succ_c i_2 \succ_b \left\{i_1, i_4\right\}$, which is a strong priority reversal, contradiction. □

For any $i \in N$, define $\mathscr{U}(i) = \{a \in H : i \succ_a N \setminus \{i\}, j \sim_a k, \forall j, k \in N \setminus \{i\}\}$, and $\mathscr{D}(i) = \{a \in H : N \setminus \{i\} \succ_a i, j \sim_a k, \forall j, k \in N \setminus \{i\}\}$.

If $H = I_{N,H}$, then $\{N, H, \succeq_{N,H}\}$ has a house allocation structure.

If $H \neq I_{N,H}$ and for some $a \in H \setminus I_{N,H}$, $|A_a^1| \geqslant 2$, then $|A_a^2| = 1$ by Claim 2. Clearly $|N| \geqslant 3$. Suppose $i \in A_a^2$, then $a \in \mathscr{D}(i) \neq \phi$. For any $j \in N, j \neq i$, by (i) of Claim 1 there exists some b such that $i \succ_b j$, thus $i \in A_b^1, j \in A_b^2$. Then weak non-reversal implies that $k \in A_b^1$ for any $k \in N \setminus \{i, j\}$. Thus $b \in \mathscr{D}(j) \neq \phi$. It remains to show that $H = I_{N,H} \cup \{\cup_{l \in N} \mathscr{D}(l)\}$. Now suppose for some $c \in H$, $c \notin \mathscr{D}(l)$ for any $l \in N$ and $c \notin I_{N,H}$, then $A_c^1 \neq \phi, |A_c^2| \geqslant 2$. Let $k_1 \in A_c^1, \{k_2, k_3\} \subseteq A_c^2$, then $\{k_2, k_3\} \succ_d k_1 \succ_c \{k_2, k_3\}$ for some $d \in \mathscr{D}(k_1)$, which is a strong priority reversal, contradiction. Hence $\{N, H, \succeq_{N,H}\}$ has an IT structure.

If $H \neq I_{N,H}$ and for any $a \in H \setminus I_{N,H}$, $|A_a^1| = 1$, then for each $a \in H \setminus I_{N,H}$, $a \in \mathscr{U}(i)$ for some $i \in N$. Thus $H = I_{N,H} \cup \{\cup_{i \in N} \mathscr{U}(i)\}$, and $|N| \geqslant 2$. It remains to show that $\mathscr{U}(i)$ is nonempty for any $i \in N$ to establish the housing market structure. $H \neq I_{N,H}$ implies that there exists some $k \in N$ with $\mathscr{U}(k) \neq \phi$. If for some $j, \mathscr{U}(j) = \phi$, then clearly $k \succ \succeq_H j$, contradicting to Claim 1. Thus $\{N, H, \succeq_{N,H}\}$ has a housing market structure. □

Proof of Lemma 4. Let $\{N, H, \succeq_{N,H}\} \in \mathscr{P}^{IT}$. Denote the assignment from the serial dictatorship with respect to σ and $R \in \mathscr{R}_H^N$ as $f^{SD}(\sigma, R)$. We will use the fact that serial dictatorships are efficient, and for any $R \in \mathscr{R}_H^N$ and an efficient μ, there exists σ such that $f^{SD}(\sigma, R) = \mu$ (Svensson, 1994). Given any $R \in \mathscr{R}_H^N$ and σ, by the construction of the queue rule, there exists σ_1 such that $f^{SD}(\sigma_1, R) = q(\sigma, R)$, so $q(\sigma, R)$ is efficient, hence individually rational and nonwasteful. Priorities cannot be violated by the queue rule: for any $i, j \in N$, $j \succ_{q_i(\sigma,R)} i$ implies $q_i(\sigma, R) \in \mathscr{D}(i)$, so either i is assigned after j, or i and j are assigned simultaneously when a loop forms, thus j does not envy i's assignment, i.e., $q_j(\sigma, R) R_j q_i(\sigma, R)$. Hence $q(\sigma, R)$ is stable. On the other hand, given $R \in \mathscr{R}_H^N$, suppose μ is stable and efficient. Then there exists σ_2 such that $f^{SD}(\sigma_2, R) = \mu$. Let $\{i : \mu(i) \notin \mathscr{D}(i)\} = B_1, \{i : \mu(i) \in \mathscr{D}(i)\} = B_2$. Construct σ_3 such that $\sigma_3^{-1}(i) < \sigma_3^{-1}(j)$ if $i \in B_1, j \in B_2$, or $i, j \in B_k, k = 1, 2$, $\sigma_2^{-1}(i) < \sigma_2^{-1}(j)$. Suppose $f^{SD}(\sigma_2, R) \neq f^{SD}(\sigma_3, R)$, then by the efficiency of serial dictatorships there exist $i, j \in N$ such that $f_i^{SD}(\sigma_3, R) P_i f_i^{SD}(\sigma_2, R)$ and $f_j^{SD}(\sigma_2, R) = f_i^{SD}(\sigma_3, R)$. Then $\sigma_2^{-1}(j) < \sigma_2^{-1}(i)$, and the stability of μ implies $j \in B_1$. Thus by the construction of $\sigma_3, \sigma_3^{-1}(j) < \sigma_3^{-1}(i)$, so $f_j^{SD}(\sigma_2, R) = f_i^{SD}(\sigma_3, R)$ implies that $f_j^{SD}(\sigma_3, R) P_j f_j^{SD}(\sigma_2, R)$. Since the problem is finite, continue in this fashion there exists a sequence $(i_1, i_2, ..., i_n)$, where $f_{i_{k+1}}^{SD}(\sigma_2, R) P_{i_k} f_{i_k}^{SD}(\sigma_2, R)$ for $k \in \{1, 2, ..., n-1\}$, and $f_{i_1}^{SD}(\sigma_2, R) P_{i_n} f_{i_n}^{SD}(\sigma_2, R)$, contradicting to μ being efficient. Hence, $f^{SD}(\sigma_3, R) = f^{SD}(\sigma_2, R) = \mu$. By the construction of σ_3, clearly $q_i(\sigma_3, R) = f_i^{SD}(\sigma_3, R)$ for $i \in B_1$. By the stability of $f^{SD}(\sigma_3, R)$, $f_i^{SD}(\sigma_3, R) P_i f_j^{SD}(\sigma_3, R)$ for any $i, j \in B_2, i \neq j$, then $f_i^{SD}(\sigma_3, R)$ is agent i's favorite house in $H \setminus \mu(B_1)$ for all $i \in B_2$. Thus a loop forms after agents in B_1 are assigned and $q_i(\sigma_3,R) = f_i^{SD}(\sigma_3,R)$ for $i \in B_2$. Therefore, $q(\sigma_3,R) = f^{SD}(\sigma_3,R) = f^{SD}(\sigma_2,R) = \mu$. $\square$

Proof of Proposition 1. Suppose $\succeq$ satisfies weak non-reversal. Given any σ and $R \in \mathscr{R}_{\mathscr{H}}^{\mathscr{N}}$, the efficiency (as well as individual rationality and nonwastefulness) of $f^{\succeq}(\sigma,R)$ follows directly from the efficiency of queue rules. At any step k of the iteration, μ_k is stable for $\left\{N_k, H_k, \succeq_{N_k,H_k}\right\}$. Suppose for some $i,j \in \mathscr{N}$, $j \succ_{f_i^{\succeq}(\sigma,R)} i$. Then clearly $f_j^{\succeq}(\sigma,R) R_j f_i^{\succeq}(\sigma,R)$ if $i,j \in N_k$ for some k. If $i \in N_s, j \in N_t$ and $s \neq t$, then $t < s$, and the nonwastefulness of μ_t implies $f_j^{\succeq}(\sigma,R) R_j f_i^{\succeq}(\sigma,R)$. This shows that $f^{\succeq}(\sigma,R)$ respects the priorities, hence $f^{\succeq}(\sigma,R)$ is stable. $\square$

Proof of Theorem 2. "if part" follows directly from Proposition 1.
"only if" part. Suppose there exists a strong priority reversal $\{i,j\} \succ_a k \succ_b \{i,j\}$. Consider the following preference profile R:

$R_i : b, a, i,$
$R_j : b, a, j,$
$R_k : a, b, k,$
$R_l : l, \forall l \in \mathscr{N} \setminus \{i,j,k\}.$

Given any stable assignment μ, $\mu(l) = l$ for all $l \in \mathscr{N} \setminus \{i,j,k\}$. By the same argument in Example 1, $\mu(k) = b$, and either $\mu(i) = a$ or $\mu(j) = a$, so μ is inefficient. Thus there does not exist a stable and efficient rule. $\square$

Proof of Theorem 3. ***(ii)*** $\Rightarrow$ ***(i)*** and ***(iv)*** $\Rightarrow$ ***(iii)*** are obvious. We first show ***(i)*** $\Rightarrow$ ***(iv)***, then ***(iii)*** $\Rightarrow$ ***(ii)***. Suppose there exists a stable, efficient and strategy-proof rule f. Then by Theorem 2, $\succeq$ satisfies weak non-reversal. Suppose that non-reversal is not satisfied, then there exist some distinct $i,j,k,l \in \mathscr{N}$ and some distinct $a,b,c,d \in \mathscr{H}$ such that $\{i,j,k\} \succ_a l$, $l \succ_b i$, $l \succ_c j$ and $l \succ_d k$. By weak non-reversal, we have $\left\{N = \{i,j,k,l\}, H = \{a,b,c,d\}, \succeq_{N,H}\right\} \in \mathscr{P}^{IT}$, with $a \in \mathscr{D}(l), b \in \mathscr{D}(i), c \in \mathscr{D}(j), d \in \mathscr{D}(k)$. It is sufficient to consider this subproblem only, since we can restrict attention to the preference profiles in which $mR_m h$ for all $m \notin \{i,j,k,l\}$ and $h \in \mathscr{H}$, and each house in $\mathscr{H} \setminus H$ is unacceptable to the agents in N. Construct the following preferences:

$R_l : a, b, c, d, l,$
$R'_l : b, a, c, d, l,$
$R_i : b, c, a, i,$

$R_j : c, b, d, j,$
$R_k : d, c, b, a, k,$
$R'_k : c, d, b, a, k.$

And consider the following three preference profiles:

$R^1 = (R_l, R_i, R_j, R'_k),$
$R^2 = (R'_l, R_i, R_j, R'_k),$
$R^3 = (R'_l, R_i, R_j, R_k).$

For any σ, $(q_l(\sigma, R^1), q_i(\sigma, R^1), q_j(\sigma, R^1), q_k(\sigma, R^1)) = (d, a, b, c)$. By Lemma 4 this is the unique stable and efficient assignment for R^1, so $\left(f_l(R^1), f_i(R^1), f_j(R^1), f_k(R^1)\right) = (d, a, b, c)$. Then strategy-proofness requires $f_l(R^2) = d$. The stability of f implies $f_j(R^2) \neq c$, since otherwise k's priority for c is violated. Then $f_i(R^2) \neq b$, since otherwise j's priority for b is violated. Finally, by efficiency it can be easily seen that $f(R^1) = f(R^2)$. Now consider R^3, again by Lemma 4 there exists a unique stable and efficient assignment: $\left(f_l(R^3), f_i(R^3), f_j(R^3), f_k(R^3)\right) = (b, c, d, a)$. But comparing $f(R^2)$ and $f(R^3)$, we have $f_k(R'_k, R^3_{-k}) P_k f_k(R_k, R^3_{-k})$, contradicting to f being strategy-proof.

(iii)* $\Rightarrow$ *(ii)*.** It is sufficient to show that there exists a stable, efficient and weakly group strategy-proof rule for any IT structure with three agents, since we can combine this rule with serial dictatorships and TTC to obtain modified priority set rules which obviously also satisfy these three axioms. Consider any $\left\{N, H, \succeq_{N,H}\right\} \in \mathscr{P}^{IT}$ with $|N| = 3$. Given an ordering σ, define the following ***$\mathscr{D}$-tiebreaking rule for DA:

(i) Suppose some house is choosing between two agents i and j with equal priority. If the third agent k is on the waiting list of (or is applying to) some other house a, then i is rejected if $a \in \mathscr{D}(i)$, j is rejected if $a \in \mathscr{D}(j)$. If $a \in \mathscr{D}(k) \cup I_{N,H}$, or k has already been rejected by all of her acceptable houses, then break the tie according to σ, e.g., j is rejected when $\sigma^{-1}(i) < \sigma^{-1}(j)$.
(ii) If the three agents apply to some $a \notin I_{N,H}$ at the same time, then one agent is rejected based on the strict priority first and let the rejected agent apply to her next choice. Break the tie between the other two agents according to (i). Similarly, if the three agents apply to some $a \in I_{N,H}$ at the same time, then one agent is rejected first according to σ and let the rejected agent apply to her next choice. Break the tie between the other two agents according to (i).

DA with such a tiebreaking rule is denoted as $f^{DA(\mathscr{D})}(\sigma, \cdot)$. By construction $f^{DA(\mathscr{D})}(\sigma, \cdot)$ is stable for any σ. Given σ and a preference profile $R \in \mathscr{R}^N_H$, denote $i_1 q(a) i_2$ if at some step of

$f^{DA(\mathscr{D})}$, some house a has to choose between i_1 and i_2, and rejects i_2 in favor of i_1. By the $\mathscr{D}$-tiebreaking rule, ***q-acyclicity*** is satisfied: we cannot have three agents j_1, j_2, j_3 and two houses a_1, a_2 such that $j_1 q(a_1) j_2 q(a_2) j_3 q(a_1) j_1$.[37] For any $i \in N$, let a_i, b_i, c_i denote agent $i's$ top three choices in $H \cup \{i\}$. Let $N = \{i, j, k\}$ and $f^{DA(\mathscr{D})}(\sigma, R) = \mu$. We first show that μ is efficient and no single agent has an incentive to misrepresent preferences at the arbitrary preference profile R. For simplicity we use $q(a)$ to denote the relation defined under R (the true preference profile), while $\widetilde{q}(a)$ is defined with respect to a manipulated preference profile under consideration. There are three cases to consider.[38]

Case 1. $\left|\left\{a_i, a_j, a_k\right\}\right| = 3$. Every agent is assigned her top choice, so μ is efficient and no agent has an incentive to misrepresent her preferences.

Case 2. $a_i = a_j \neq a_k$. Without loss of generality, suppose $iq(a_i)j$. Then q-acyclicity implies that we cannot have $jq(a_k)kq(a_i)i$, thus $\mu(i) = a_i$. Clearly i has no incentive to misrepresent preferences.

If $b_j \neq a_k$, then $\mu(k) = a_k, \mu(j) = b_j$. μ is efficient and only j could potentially manipulate to obtain her first choice a_i. j cannot be assigned a_i by applying to any house other than a_k in the first step, or by first applying to a_k and getting rejected. If j first applies to a_k and $j\widetilde{q}(a_k)k$, then q-acyclicity implies $\mu(j) = a_k$, so such a manipulation strategy cannot be successful.

If $b_j = a_k$ and $kq(a_k)j$, then $\mu(k) = a_k$ and $\mu(j) = c_j$. μ is efficient. j will always be rejected by a_i and a_k for any reported preferences.

If $b_j = a_k$ and $jq(b_j)k$, then by q-acyclicity $\mu(j) = b_j, \mu(k) \notin \left\{a_i, b_j\right\}$. μ is efficient. j cannot obtain a_i no matter which house she applies to in the first step. We now consider k's incentives. There are only two potentially successful manipulation strategies for k: applying to a house $h \notin \left\{a_i, b_j\right\}$ in the first step to alter the tiebreaking between i and j (so we must have $i \sim_{a_i} j$ in this situation), and "pooling" with i, j, i.e., applying to a_i first. Suppose the first manipulation strategy is successful, then $i \sim_{a_i} j$, and $j\widetilde{q}(a_i)i, i\widetilde{q}(h)k$, thus $h \notin \mathscr{D}(i)$ and $\sigma^{-1}(j) < \sigma^{-1}(i)$. Then from $iq(a_i)j$ we have $b_j \in \mathscr{D}(j)$, but this contradicts to $jq(b_j)k$. Now

[37] If q-acyclicity is violated, then there exists a *rejection cycle*: a_1 rejects j_2 in favor of j_1, then j_2 applies to a_2 and a_2 rejects j_3 in favor of j_2, finally j_3 applies to a_1 and j_1 gets rejected. To see that such a rejection cycle does not exist, i.e., $j_1 q(a_1) j_2 q(a_2) j_3$ implies that we cannot have $j_3 q(a_1) j_1$, consider the following three cases. Case 1: $a_1 \in \mathscr{D}(j_2)$. By the $\mathscr{D}$-tiebreaking rule, $j_2 q(a_2) j_3$ implies $j_2 \succ_{a_2} j_3$, thus $a_2 \in \mathscr{D}(j_3)$ and it is not possible that $j_3 q(a_1) j_1$. Case 2: $a_1 \in I_{N,H}$. $j_2 q(a_2) j_3$ implies $a_2 \notin \mathscr{D}(j_2)$. $j_1 q(a_1) j_2$ implies $a_2 \notin \mathscr{D}(j_1)$. Thus we have either $a_2 \in \mathscr{D}(j_3)$ or $a_2 \in I_{N,H}$. If $a_2 \in \mathscr{D}(j_3)$, then clearly it is not possible that $j_3 q(a_1) j_1$. If $a_2 \in I_{N,H}$, then $j_1 q(a_1) j_2 q(a_2) j_3$ implies $\sigma^{-1}(j_1) < \sigma^{-1}(j_2) < \sigma^{-1}(j_3)$, thus we cannot have $j_3 q(a_1) j_1$. Case 3: $a_1 \in \mathscr{D}(j_3)$. Clearly it is not possible that $j_3 q(a_1) j_1$.

[38] If for some $l \in N$, $a_l = l$, then $f^{DA(\mathscr{D})}(\sigma, R)$ is obviously efficient, and no agent can manipulate at R. So we restrict attention to the cases in which $a_l \in H$ for all l in N.

assume that the second manipulation strategy is successful, then k cannot be the first rejected agent when all the three agents apply to a_i first. If i is the first rejected agent, then $a_i \in I_{N,H}$ and $\sigma^{-1}(j) < \sigma^{-1}(i)$. So $iq(a_i)j$ implies $b_j \in \mathscr{D}(j)$, contradicting to $jq(b_j)k$. If j is the first rejected agent, then i must be the second rejected agent for such a manipulation strategy to be successful for k. This implies $i \sim_{a_i} k$ and $b_j \notin \mathscr{D}(k)$. Then $jq(b_j)k$ implies $\sigma^{-1}(j) < \sigma^{-1}(k)$ and $a_i \notin \mathscr{D}(j)$, contradicting to j being the first rejected agent when all the three agents apply to a_i first.

Case 3. $a_i = a_j = a_k$. Without loss of generality suppose k is the first rejected agent and j is the second rejected agent, then $\mu(i) = a_i$ and at least one of j and k is assigned her second choice. Clearly μ is efficient. q-acyclicity implies that neither j nor k can obtain a_i, regardless of the reported preferences. Thus we are only left to show that if j or k is assigned her third choice (so $b_j = b_k$), then she cannot manipulate to obtain her second choice. If $\mu(k) = c_k$, the only potentially successful manipulation strategy for k is to apply to some house $h \notin \{a_i, b_k\}$ to alter the tiebreaking between i and j at a_i, but by the same argument in Case 2 such a strategy cannot be successful. If $\mu(j) = c_j$, similarly the only possible manipulation for j is to apply to some house $h \neq a_i$ first such that $k\widetilde{q}(a_i)i$. Then it must be the case that $a_i \in I_{N,H}$ and $h \in \mathscr{D}(i)$, then j is assigned h under the manipulated preference profile. If such a manipulation is successful, i.e., j obtains her second choice, then $b_j = h \in \mathscr{D}(i)$, contradicting to $iq(a_i)j$.

It remains to show that $f^{DA(\mathscr{D})}$ is weakly group strategy-proof. Suppose two agents can jointly manipulate and become strictly better-off, then it is sufficient to consider the following three cases: *(i)* $a_i = a_j \neq a_k, b_j = a_k, iq(a_i)j, jq(b_j)k$; *(ii)* $a_i = a_j = a_k, b_j = b_k$, k is the first rejected agent, then $iq(a_i)j, jq(b_j)k$; *(iii)* $a_i = a_j = a_k, b_j = b_k$, j is the first rejected agent, then $iq(a_i)k, jq(b_j)k$. In each case, $\mu(j) = b_j$. It can be easily shown that in order for j to obtain a_i, j must form tie with i at a_i, while k applies to some other house $h \in \mathscr{D}(i)$ to influence the tiebreaking such that i is rejected. But this would imply that k is assigned h and cannot be strictly better-off. □

Proof of Theorem 4. ***(iii)*** $\Rightarrow$ ***(ii)*** is obvious. ***(ii)*** $\Rightarrow$ ***(i)***. Since both serial dictatorships and TTC are strategy-proof and nonbossy, when $\mathscr{P}^m = \mathscr{P}^{HA} \cup \mathscr{P}^{HM}$, $f^{\succeq}(\sigma, \cdot)$ is strategy-proof and nonbossy, hence group strategy-proof by Lemma 1. $f^{\succeq}(\sigma, \cdot)$ is stable and efficient by Proposition 1.

(i) $\Rightarrow$ ***(iii)*.** Suppose there exists a stable, efficient and group strategy-proof rule f. By Theorem 2, $\succeq$ satisfies weak non-reversal. Assume *(iii)* is not true, then there exist three distinct agents $\{1,2,3\}$ and three distinct houses $\{a,b,c\}$ such that $\{2,3\} \succ_a 1, 1 \succ_b 2$ and $1 \succ_c 3$. Since

$|\mathscr{H}| > 3$, there exists some $d \notin \{a,b,c\}$, and $\left\{N = \{1,2,3\}, H = \{a,b,c,d\}, \succeq_{N,H}\right\}$ is minimal. Then weak non-reversal implies $\left\{N, H, \succeq_{N,H}\right\} \in \mathscr{P}^{IT}$, where $a \in \mathscr{D}(1)$, $b \in \mathscr{D}(2)$ and $c \in \mathscr{D}(3)$. It is sufficient to consider this subproblem only. The following result, from Lemma 1 of Svensson (1999), will be helpful for our proof. It states that group strategy-proofness implies *Maskin monotonicity*.

Claim 3 *(Svensson, 1999). Suppose $\bar{f}$ is a group strategy-proof rule for some $\left\{\bar{N}, \bar{H}, \succeq_{\bar{N},\bar{H}}\right\} \in \mathscr{P}$. Given any $R, R' \in \mathscr{R}_{\bar{H}}^{\bar{N}}$, if for any $i \in \bar{N}$, $a \in \bar{H} \cup \{i\}$, $\bar{f}_i(R) R_i a$ implies $\bar{f}_i(R) R'_i a$, then $\bar{f}(R) = \bar{f}(R')$.*

We consider two possible cases.

Case 1. $d \in \mathscr{D}(i)$ for some $i \in \{1,2,3\}$. Without loss of generality, suppose $d \in \mathscr{D}(3)$.

Step 1. $f(c,d \mid c,d \mid d,a) = (d,c,a), f(c,d \mid c,d \mid d,b) = (c,d,b)$.[39]

By Lemma 4, $f(c,d \mid c,d \mid d,a) \in \{(c,d,a),(d,c,a)\}$. First we want to show $f(c,d \mid c,d \mid d,a) = (d,c,a)$. Again, by Lemma 4, $f(a,c,b \mid c,a \mid c,a) = (b,c,a)$, then by Claim 3, $f(c,a,b \mid c,d \mid c,a) = (b,c,a)$. So if $f(c,d \mid c,d \mid d,a) = (c,d,a)$, then given the true preference profile $(c,a,b \mid c,d \mid c,a)$, agent 1 and agent 3 can jointly manipulate to $(c,d \mid c,d \mid d,a)$ such that 1 is strictly better-off while 3 gets the same house. Thus $f(c,d \mid c,d \mid d,a) = (d,c,a)$. By a similar argument, it can be shown that $f(c,d \mid c,d \mid d,b) = (c,d,b)$.[40]

Step 2. *$f(c,d \mid c,d \mid d,a) = (d,c,a)$ implies $f(c,1 \mid c,2 \mid d,3) = (1,c,d)$, and $f(c,d \mid c,d \mid d,b) = (c,d,b)$ implies $f(c,1 \mid c,2 \mid d,3) = (c,2,d)$. Hence, such a stable, efficient and group strategy-proof rule f does not exist.*

Given $f(c,d \mid c,d \mid d,a) = (d,c,a)$, strategy-proofness and efficiency imply $f(c,1 \mid c,d \mid d,a) = (1,c,d)$. Then by Claim 3, $f(c,1 \mid c,2 \mid d,3) = (1,c,d)$. Similarly, $f(c,d \mid c,d \mid d,b) = (c,d,b) \Rightarrow f(c,d \mid c,2 \mid d,b) = (c,2,d) \Rightarrow f(c,1 \mid c,2 \mid d,3) = (c,2,d)$. A contradiction is reached since two different assignments are established for the preference profile $(c,1 \mid c,2 \mid d,3)$.

Case 2. $d \in I_{N,H}$, i.e., $1 \sim_d 2 \sim_d 3$.

Step 1. $f(d,a,b \mid d,a \mid d,a) \in \{(b,d,a),(b,a,d)\}$.

By Lemma 4, $f(a,d,b \mid d,a \mid d,a) \in \{(b,d,a),(b,a,d)\}$, then Claim 3 implies $f(d,a,b \mid d,a \mid d,a) \in \{(b,d,a),(b,a,d)\}$.

[39] For simplicity, we denote $f(R_1 : cR_1d; R_2 : cR_2d; R_3 : dR_3a)$ as $f(c,d \mid c,d \mid d,a)$, and unacceptable houses are not listed.

[40] Specifically, first, by Lemma 4, $f(c,d \mid c,d \mid d,b) \in \{(c,d,b),(d,c,b)\}$. Again by Lemma 4, $f(c,b \mid b,c,a \mid c,b) = (c,a,b)$, which implies $f(c,d \mid c,b,a \mid c,b) = (c,a,b)$ by Claim 3. Then if $f(c,d \mid c,d \mid d,b) = (d,c,b)$, given the true preference profile $(c,d \mid c,b,a \mid c,b)$, agent 2 and agent 3 can jointly manipulate.

Step 2. *$f(d,a,b \mid d,a \mid d,a)=(b,d,a)$ implies $f(d,b \mid d,b,a \mid d,b)=(d,a,b)$.*

By Claim 3, $f(d,a,b \mid d,a \mid d,a)=(b,d,a)$ implies $f(d,b,a \mid d,b,a \mid d,a,b)=(b,d,a)$, then strategy-proofness implies $f_3(d,b,a \mid d,b,a \mid d,b,a)\neq d$. By Lemma 4, $f_2(d,b,a \mid b,d,a \mid d,b,a)=a$, then $f_2(d,b,a \mid d,b,a \mid d,b,a)=a$ by strategy-proofness. Combined with $f_3(d,b,a \mid d,b,a \mid d,b,a)\neq d$ we have $f(d,b,a \mid d,b,a \mid d,b,a)=(d,a,b)$ by efficiency. It follows that $f(d,b \mid d,b,a \mid d,b)=(d,a,b)$.

Step 3. *(i) $f(d,a,b \mid d,a \mid d,a)=(b,d,a)$ implies $f(d,a \mid d,b \mid d,c)=(a,d,c)$; (ii) $f(d,b \mid d,b,a \mid d,b)=(d,a,b)$ implies $f(d,a \mid d,b \mid d,c)=(d,b,c)$.*

Given $f(d,a,b \mid d,a \mid d,a)=(b,d,a)$, by Claim 3, $f(d,a,b \mid d,c \mid d,a)=(b,d,a)$, then $f_3(d,a,b \mid d,c \mid d,c,a)\in\{c,a\}$ by strategy-proofness. If $f_3(d,a,b \mid d,c \mid d,c,a)=a$, then nonbossiness implies $f(d,a,b \mid d,c \mid d,c,a)=(b,d,a)$, so c is wasted. Thus $f_3(d,a,b \mid d,c \mid d,c,a)=c$. Since $2\succ_c 3$, stability implies $f_2(d,a,b \mid d,c \mid d,c,a)=d$. So we have $f(d,a,b \mid d,c \mid d,c,a)=(a,d,c)$, then by Claim 3, $f(d,a \mid d,b \mid d,c)=(a,d,c)$.

$f(d,b \mid d,b,a \mid d,b)=(d,a,b)$ implies $f(d,a \mid d,b \mid d,c)=(d,b,c)$ by a similar argument.[41]

Step 4. *$f(d,a,b \mid d,a \mid d,a)=(b,a,d)$, $f(d,b \mid d,b,c \mid d,b)=(d,c,b)$, $f(d,c \mid d,c \mid d,c,a)=(c,d,a)$.*[42]

Step 2 and step 3 imply that we cannot have $f(d,a,b \mid d,a \mid d,a)=(b,d,a)$. Then by step 1, $f(d,a,b \mid d,a \mid d,a)=(b,a,d)$. By a set of symmetric arguments, it can be shown that $f(d,b \mid d,b,c \mid d,b)=(d,c,b)$, $f(d,c \mid d,c \mid d,c,a)=(c,d,a)$.

Step 5. *Such a stable, efficient and group strategy-proof rule f does not exist.*

Consider the following preference profile R:

$R_1 : d,b,a,$

$R_2 : d,c,b,$

$R_3 : d,a,c.$

By step 4, $f(d,c \mid d,c \mid d,c,a)=(c,d,a)$, thus by Claim 3, $f(d,c \mid d,c,b \mid d,a,c)=(c,d,a)$. Then $f_1(R)\neq d$ by strategy-proofness. Similarly, $f(d,a,b \mid d,a \mid d,a)=(b,a,d)\Rightarrow f(d,b,a \mid d,a \mid d,a,c)=(b,a,d)\Rightarrow f_2(R)\neq d$. And $f(d,b \mid d,b,c \mid d,b)=(d,c,b)\Rightarrow f(d,b,a \mid d,c,b \mid$

[41]Specifically, $f(d,b \mid d,b,a \mid d,b)=(d,a,b)\Rightarrow f(d,c \mid d,b,a \mid d,b)=(d,a,b)\Rightarrow f(d,c \mid d,b,a \mid d,c,b)=(d,b,c)\Rightarrow f(d,a \mid d,b \mid d,c)=(d,b,c)$.

[42]The idea is similar to the preference-based tiebreaking rule in the proof of Theorem 3. When all the agents have the same first choice d and the same second choice $h\in\{a,b,c\}$, with $h\in\mathscr{D}(i)$, then i will be assigned her third choice. The assignment of d depends on i's third choice: if i's third choice is in $\mathscr{D}(j)$, $j\neq i$, then the third agent k will be assigned d.

$d,b) = (d,c,b) \Rightarrow f_3(R) \neq d$. Nonwastefulness is violated, hence such a rule f does not exist. $\square$

Proof of Corollary 1. "only if" part follows from Theorem 4 and the definition of housing market structures in Lemma 3.

"if" part. Suppose that any $\{N, H, \succeq_{N,H}\} \in \mathscr{P}^m$ with $|N| = |H|$ is either a house allocation problem or a housing market problem. Assume to the contrary, there does not exist a stable, efficient and group strategy-proof rule, then by Theorem 4 there exists a weak priority reversal: $\{i,j\} \succ_a k, k \succ_b i$ and $k \succ_c j$. If $b = c$, pick any $h \in \mathscr{H} \setminus \{a,b\}$. Then $\{N = \{i,j,k\}, H = \{a,b,h\}, \succeq_{N,H}\} \in \mathscr{P}^m$, but it is not a house allocation problem or a housing market problem. If $b \neq c$, then $\{N' = \{i,j,k\}, H' = \{a,b,c\}, \succeq_{N',H'}\} \in \mathscr{P}^m$ and it is not a house allocation problem or a housing market problem either, contradiction. $\square$

Proof of Proposition 2. We will need the following mathematical result discussed in Roth and Sotomayor (1990) (Lemma 2.30).

Lemma 5. *Let f and g be functions from a finite set X into a set Y where f is a one-to-one correspondence. Then there is a nonempty subset A of X such that f and g map A onto $f(A)$.*

Suppose that μ is stable, ν Pareto dominates μ, and for any $i \in \mathscr{N}$ with $\mu(i) = \nu(i)$, there does not exist $j \in \mathscr{N}, a \in \mathscr{H}$ such that $i \succ_a j, \nu(j) = a$ and $aP_i\nu(i)$. Let $N_P = \{i \in \mathscr{N} : \nu(i)P_i\mu(i)\}$. Recall that Pareto-improving a stable assignment is only possible through reshuffling the houses among the agents, so $\mu(N_P) = \nu(N_P) \subseteq \mathscr{H}$. Consider any $a \in \mu(N_P)$. There exists some agent $i \in N_P$ such that $aP_i\mu(i)$ and $i \succeq_a j$ for all $j \in N_P$ with $aP_j\mu(j)$. Pick such an agent for a and denote her as $t(a)$. Moreover, if some $j \notin N_P$ and $aP_j\mu(j)$, then $aP_j\nu(j)$. By assumption, we have $\nu^{-1}(a) \succeq_a j$. Since $\nu^{-1}(a) \in N_p$, $t(a) \succeq_a \nu^{-1}(a) \succeq_a j$. In sum, $t(a) \succeq_a k$ for all $k \in \mathscr{N}$ with $aP_k\mu(k)$.

t maps $\mu(N_P)$ into N_P, and μ^{-1} maps $\mu(N_P)$ onto N_P in a one-to-one fashion. By Lemma 5, there exists some nonempty $H \subseteq \mu(N_P)$ such that t maps H onto $\mu^{-1}(H)$ in a one-to-one fashion. Let $N_C = \mu^{-1}(H)$.

Construct an assignment μ' as follows: if $i \in N_C, a \in H$ and $t(a) = i$, then $\mu'(i) = a$, $\mu'(i) = \mu(i)$ otherwise. Then μ' Pareto dominates μ. To complete the proof, it remains to show that μ' is stable. Assume to the contrary, there exist some $i,j \in \mathscr{N}$ and $a \in \mathscr{H}$ such that $i \succ_a j$, $\mu'(j) = a$ and $aP_i\mu'(i)$. Clearly $aP_i\mu(i)$. So the stability of μ implies that $\mu(j) \neq a$. Then

$j \in N_C$, $a \in H$ and $t(a) = j$. A contradiction is reached since $t(a) \succeq_a k$ for all $k \in \mathcal{N}$ with $aP_k\mu(k)$.

Finally, notice that μ' is constructed by letting the agents in N_C exchange their assignments at μ, and each exchange cycle among N_C corresponds to a stable improvement cycle (Erdil and Ergin, 2008). $\square$

Proof of Proposition 3. "if" part. Suppose that there exists a strong priority reversal $\{i,j\} \succ_a k \succ_b \{i,j\}$. If $i \sim_b j$, then this is a cyclic tie and $\succeq$ is not EW-acyclic. If $i \succ_b j$, there is an Ergin-cycle $k \succ_b i \succ_b j \succ_a k$. If $j \succ_b i$, there is an Ergin-cycle $k \succ_b j \succ_b i \succ_a k$. So $\succeq$ is not Ergin-acyclic when it is not $i \sim_b j$.

"only if" part. We have shown that weak non-reversal implies Ergin-acyclicity. Suppose $\succeq$ satisfies weak non-reversal, but assume to the contrary there exists a cyclic tie $i_1 \sim_a i_2$ between two distinct agents i_1, i_2. Then there exist $j_1, j_2 \in \mathcal{N} \setminus \{i_1, i_2\}$ and $b_1, b_2 \in \mathcal{H}$ such that we have four possible cases:

Case 1. $j_1 \neq j_2, b_1 \neq b_2$ and $\{j_1, j_2\} \succ_a i_1 \sim_a i_2, i_1 \succ_{b_1} j_1, i_2 \succ_{b_2} j_2$.
Then $\left\{N = \{i_1, i_2, j_1, j_2\}, H = \{a, b_1, b_2\}, \succeq_{N,H}\right\} \in \mathcal{P}^m$, contradicting to Lemma 3.

Case 2. $j_1 \neq j_2, b_1 = b_2$ and $\{j_1, j_2\} \succ_a i_1 \sim_a i_2, i_1 \succ_{b_1} j_1, i_2 \succ_{b_1} j_2$. We have $j_2 \succeq_{b_1} i_1$, because otherwise $\{j_1, j_2\} \succ_a i_1 \succ_{b_1} \{j_1, j_2\}$. But then $\{i_1, i_2\} \succ_{b_1} j_1 \succ_a \{i_1, i_2\}$, contradiction.

Case 3. $j_1 = j_2, b_1 = b_2$ and $j_1 \succ_a i_1 \sim_a i_2, \{i_1, i_2\} \succ_{b_1} j_1$. This is a strong priority reversal, contradiction.

Case 4. $\{i_1, i_2\} \succ_{b_1} j_1, j_1 \succ_{b_2} j_2, j_2 \succ_a i_1 \sim_a i_2$. First it can be easily verified that $|\{a, b_1, b_2\}| = 3$, since otherwise there exists an Ergin-cycle. Then $\left\{N = \{i_1, i_2, j_1, j_2\}, H = \{a, b_1, b_2\}, \succeq_{N,H}\right\} \in \mathcal{P}^m$, contradicting to Lemma 3. $\square$

Proof of Theorem 5. ***(ii)*$\Rightarrow$ *(i)*** is obvious. ***(i)*$\Rightarrow$ *(iii)*** follows from the discussion below Example 4.

(iii)*$\Rightarrow$ *(ii). Suppose *(iii)* is satisfied. Let A be the smallest priority set for $\{\mathcal{N}, \mathcal{H}, \succeq\}$. As discussed in Section 5.5, $\left\{A, \mathcal{H}, \succeq_{A,\mathcal{H}}\right\}$ is a house allocation or a housing market structure, and when it is a house allocation structure, there exists a stable, efficient and strategy-proof rule. To show that such a rule also exists if $\left\{A, \mathcal{H}, \succeq_{A,\mathcal{H}}\right\}$ is a housing market structure, we first describe the top cycles rules.

For ease of exposition, we abuse the notations a bit and consider a housing market problem $\left\{\bar{\mathcal{N}}, \bar{\mathcal{H}}, \omega\right\}$, where $|\bar{\mathcal{N}}| = |\bar{\mathcal{H}}|$ and $\omega : \bar{\mathcal{N}} \to \bar{\mathcal{H}}$ is a bijection that specifies the initial endowments. Each $i \in \bar{\mathcal{N}}$ has a weak preference relation, $\bar{R}_i \in \bar{\mathcal{R}}^{\{i\}}_{\bar{\mathcal{H}}}$, over $\bar{\mathcal{H}}$. Given $\bar{R} \in \bar{\mathcal{R}}^{\bar{\mathcal{N}}}_{\bar{\mathcal{H}}}$, and an ordering σ of the agents, the top cycles rule is defined via the following algorithm from Jaramillo and Manjunath (2012).

Each step of the algorithm proceeds in three phases: departure, pointing, and trading. In each step, an agent is *unsatisfied* if she does not hold one of her top houses among the remaining ones.[43] In the first step, each $i \in \bar{\mathscr{N}}$ holds $\omega(i)$.

1. Departure: A group of agents "departs" if each agent in the group holds one of her top houses, and all of the top houses of the group are held by them. Once a group departs, each of them is assigned what she holds and leaves the problem with her assignment.

2. Pointing: Each remaining agent points at a remaining agent holding one of her top houses, according to the following stages:

Stage 1. For each remaining j who holds the same house that she held in the previous step, each i that pointed at j in the previous step points at j in the current step.

Stage 2. Each agent with a unique top house points at the agent holding it.

Stage 3. Each agent who has at least one of her top houses held by an unsatisfied agent points at whomever has the highest order (according to σ) among such unsatisfied agents.

Stage 4. For each agent who is not yet pointing at anyone, her top houses are all held by satisfied agents. If at least one of her top houses is held by a satisfied agent who points at an unsatisfied agent, then she points at whomever points at the unsatisfied agent with the highest order. If two or more of her satisfied "candidates" point at the unsatisfied agent with the highest order, she points at the satisfied candidate with the highest order.

Stage ... And so on.

Finally, any agent who cannot reach an unsatisfied agent points at the agent with the highest order, other than herself, holding one of her top houses.

3. Trading: There is at least one cycle of remaining agents. For each such cycle, agents trade according to the way that they point and what they hold for the next step is updated accordingly.

The algorithm terminates when every agent has departed.

Now, given $\{\mathscr{N}, \mathscr{H}, \succeq\}$, construct a corresponding housing market problem, $\left\{\bar{\mathscr{N}}, \bar{\mathscr{H}}, \omega\right\}$, as follows.

Let $\mathscr{N} \subseteq \bar{\mathscr{N}}$ and $\mathscr{H} \subseteq \bar{\mathscr{H}}$. For each $i \in \mathscr{N} \setminus A$, construct a dummy house $a^d(i)$ and let $\omega(i) = a^d(i)$. For each $a \in I_{A,\mathscr{H}}$, construct a dummy agent $i^d(a)$ and let $\omega(i^d(a)) = a$. For any

[43] In the following description of each step, an agent's top houses are referred to her top houses among the remaining ones. That is, if H is the set of remaining houses, i's top houses are those in the set $\left\{a \in H : a\bar{R}_i b, \forall b \in H\right\}$.

$i \in A$ with $|\mathscr{U}(i)| = 1$, let $\{\omega(i)\} = \mathscr{U}(i)$. If $i \in A$ and $|\mathscr{U}(i)| > 1$, arbitrarily pick some house such that $\omega(i) \in \mathscr{U}(i)$. Then for each $a \in \mathscr{U}(i) \setminus \{\omega(i)\}$, construct a dummy agent $i^d(a)$ such that $\omega(i^d(a)) = a$. Let $D = \left\{i^d(a) : a \in \mathscr{U}(i) \setminus \{\omega(i)\}, i \in A\right\}$. Then for each $i \in D$, construct a dummy agent $\kappa(i)$ and a dummy house $a^d(\kappa(i))$ such that $\omega(\kappa(i)) = a^d(\kappa(i))$. Hence we have $\bar{\mathscr{N}} = \mathscr{N} \cup \left\{i^d(a) : a \in I_{A,\mathscr{H}}\right\} \cup D \cup \{\kappa(i) : i \in D\}$, $\bar{\mathscr{H}} = \left\{\omega(i) : i \in \bar{\mathscr{N}}\right\}$ and $|\bar{\mathscr{N}}| = |\bar{\mathscr{H}}|$.

Now given any preference profile $R \in \mathscr{R}^{\mathscr{N}}_{\mathscr{H}}$, we map it to some $\bar{R} \in \bar{\mathscr{R}}^{\bar{\mathscr{N}}}_{\bar{\mathscr{H}}}$ as follows.

For each $i \in \mathscr{N}$, define $\bar{R}_i$ on $\bar{\mathscr{H}}$ such that *(1)* for any $a, b \in \mathscr{H}$ with $aP_i i$ and $bP_i i$, $a\bar{R}_i b$ if and only if $aR_i b$, and *(2)* for any $a \in \mathscr{H}$ with $aP_i i$, $b \in \mathscr{H}$ with $iR_i b$ and $c \in \bar{\mathscr{H}} \setminus \mathscr{H}$, $a\bar{P}_i b\bar{I}_i c$. For each $i \in \bar{\mathscr{N}} \setminus \{\mathscr{N} \cup D\}$, let i have degenerate preferences, i.e., $a\bar{I}_i b$ for all $a, b \in \bar{\mathscr{H}}$. For each $i \in D$, there exists $j \in A$ with $\omega(i) \in \mathscr{U}(j)$. Let $\omega(j)\bar{I}_i\omega(\kappa(i))\bar{P}_i a\bar{I}_i b$ for all $a, b \in \bar{\mathscr{H}} \setminus \{\omega(j), \omega(\kappa(i))\}$.

Pick some ordering σ of $\bar{\mathscr{N}}$ such that *(1)* for any $i \in A$, $j \in \mathscr{N} \setminus A$ and $k \in \bar{\mathscr{N}} \setminus \mathscr{N}$, $\sigma^{-1}(i) < \sigma^{-1}(j) < \sigma^{-1}(k)$, and *(2)* for any $i, j \in \mathscr{N} \setminus A$, if $i \succ_a j$ for some $a \in \mathscr{H}$, then $\sigma^{-1}(i) < \sigma^{-1}(j)$.

The top cycles rule (with respect to σ), $\bar{f}^{TC}$, for $\left\{\bar{\mathscr{N}}, \bar{\mathscr{H}}, \omega\right\}$ induces a rule, f^{TC}, for $\{\mathscr{N}, \mathscr{H}, \succeq\}$ as follows. For any $R \in \mathscr{R}^{\mathscr{N}}_{\mathscr{H}}$ and $i \in \mathscr{N}$, $f^{TC}_i(R) = \bar{f}^{TC}_i(\bar{R})$ if $\bar{f}^{TC}_i(\bar{R}) \in \mathscr{H}$ and $\bar{f}^{TC}_i(\bar{R})R_i i$, $f^{TC}_i(R) = i$ otherwise. Jaramillo and Manjunath (2012) show that $\bar{f}^{TC}$ is efficient and strategy-proof. Moreover, each agent's assignment is always weakly better than her endowment. The strategy-proofness of f^{TC} follows directly from the strategy-proofness of $\bar{f}^{TC}$ and the construction of $\bar{R}$ from R. We now show that f^{TC} is stable and efficient.

Fix some $R \in \mathscr{R}^{\mathscr{N}}_{\mathscr{H}}$. First, $f^{TC}(R)$ is individually rational by construction. Suppose that there exists some assignment μ for $\{\mathscr{N}, \mathscr{H}, \succeq\}$ and μ Pareto dominates $f^{TC}(R)$. Then there exists an assignment $\bar{\mu}$ for $\left\{\bar{\mathscr{N}}, \bar{\mathscr{H}}, \omega\right\}$ such that $\bar{\mu}(i) = \mu(i)$ for each $i \in \mathscr{N}$ with $\mu(i)P_i i$, and $\bar{\mu}(i) = \omega(\kappa(i))$ for each $i \in D$. First, for any $i \in \mathscr{N}$ with $iR_i\mu(i)$, by individual rationality, $f^{TC}_i(R)I_i i$. Then by the construction of $\bar{R}_i$ and f^{TC}, $a\bar{R}_i\bar{f}^{TC}_i(\bar{R})$ for all $a \in \bar{\mathscr{H}}$. Hence $\bar{\mu}(i)\bar{R}_i\bar{f}^{TC}_i(\bar{R})$. Second, since $\mu(i)I_i f^{TC}_i(R)$ for each $i \in \mathscr{N}$ with $iR_i\mu(i)$, we have $\mu(i)R_i f^{TC}_i(R)$ for all $i \in \mathscr{N}$ with $\mu(i)P_i i$, and $\mu(j)P_j f^{TC}_j(R)$ for some $j \in \mathscr{N}$ with $\mu(j)P_j j$. It follows that $\bar{\mu}(i)\bar{R}_i\bar{f}^{TC}_i(\bar{R})$ for all $i \in \mathscr{N}$ with $\mu(i)P_i i$, and $\bar{\mu}(j)\bar{P}_j\bar{f}^{TC}_j(\bar{R})$ for some $j \in \mathscr{N}$ with $\mu(j)P_j j$. Finally, since each $i \in D$ is assigned one of her top houses under $\bar{\mu}$ and all the other dummy agents have degenerate preferences, $\bar{\mu}$ Pareto dominates $\bar{f}^{TC}(\bar{R})$, contradicting to the efficiency of $\bar{f}^{TC}(\bar{R})$. Therefore, $f^{TC}(R)$ is efficient. It follows that $f^{TC}(R)$ is nonwasteful.

It remains to show that priorities are not violated. First, consider any $i \in A$ and $a \in \mathscr{U}(i)$. If $a = \omega(i)$, then clearly $\bar{f}^{TC}_i(\bar{R})\bar{R}_i a$ and $f^{TC}_i(R)R_i a$. If $a \neq \omega(i)$, then some $j \in D$ is initially endowed with a. j has two top houses: $\omega(i)$ and $\omega(\kappa(j))$. Since $\kappa(j)$ is a satisfied agent in any step, j always points at i if i is not satisfied yet. i can always point at j to get a, hence $\bar{f}_i^{TC}(\bar{R})\bar{R}_i a$ and $f_i^{TC}(R)R_i a$. Second, consider any $i \in A$ and $j \in \mathscr{N} \setminus A$ with $f_j^{TC}(R) \in \mathscr{H}$. Since $\sigma^{-1}(i) < \sigma^{-1}(j)$ and there does not exist any $k \in \bar{\mathscr{N}}$ with $\omega(j)\bar{P}_k\omega(i)$, no agent points at j before i becomes satisfied. Hence i does not envy j's assignment. Finally, by a similar argument, for any $i, j \in \mathscr{N} \setminus A$ and $a \in \mathscr{H}$, if $i \succ_a j$ and $f_j^{TC}(R) = a$, then $f_i^{TC}(R)R_i a$. □

References

ABDULKADIROĞLU, A., P. A. PATHAK, AND A. E. ROTH (2005a): "The New York City high school match," *American Economic Review*, 95, 364–367.

ABDULKADIROĞLU, A., P. A. PATHAK, A. E. ROTH, AND T. SÖNMEZ (2005b): "The Boston public school match," *American Economic Review*, 95, 368–371.

ABDULKADIROĞLU, A. AND T. SÖNMEZ (1999): "House allocation with existing tenants," *Journal of Economic Theory*, 88, 233–260.

——— (2003): "School choice: A mechanism design approach," *American Economic Review*, 93, 729–747.

ALCALDE, J. AND S. BARBERÀ (1994): "Top dominance and the possibility of strategy-proof stable solutions to matching problems," *Economic theory*, 4, 417–435.

ALCALDE, J. AND A. ROMERO-MEDINA (2015): "Strategy-proof fair school placement," *mimeo*.

BARBERÀ, S., D. BERGA, AND B. MORENO (2016): "Group strategy-proofness in private good economies," *American Economic Review*, 106, 1073–1099.

CANTALA, D. AND S. PÁPAI (2014): "Reasonably and securely stable matching," *mimeo*.

DUBINS, L. E. AND D. A. FREEDMAN (1981): "Machiavelli and the Gale-Shapley algorithm," *American Mathematical Monthly*, 88, 485–494.

DUR, U., A. A. GITMEZ, AND O. YILMAZ (2015): "School choice under partial fairness," *mimeo*.

EHLERS, L. (2002): "Coalitional strategy-proof house allocation," *Journal of Economic Theory*, 105, 298–317.

——— (2006): "Respecting priorities when assigning students to schools," *mimeo*.

EHLERS, L. AND A. ERDIL (2010): "Efficient assignment respecting priorities," *Journal of Economic Theory*, 145, 1269–1282.

EHLERS, L. AND B. KLAUS (2004): "Resource-monotonicity for house allocation problems," *International Journal of Game Theory*, 32, 545–560.

EHLERS, L., B. KLAUS, AND S. PÁPAI (2002): "Strategy-proofness and population-monotonicity for house allocation problems," *Journal of Mathematical Economics*, 38, 329–339.

EHLERS, L. AND T. MORRILL (2017): "(Il)legal assignments in school choice," *mimeo*.

EHLERS, L. AND A. WESTKAMP (2016): "Strategy-proof tie-breaking," *mimeo*.

ERDIL, A. AND H. ERGIN (2008): "What's the matter with tie-breaking? Improving efficiency in school choice," *American Economic Review*, 98, 669–689.

ERGIN, H. I. (2002): "Efficient resource allocation on the basis of priorities," *Econometrica*, 70, 2489–2497.

GALE, D. AND L. S. SHAPLEY (1962): "College admissions and the stability of marriage," *American Mathematical Monthly*, 69, 9–15.

HAERINGER, G. AND F. KLIJN (2009): "Constrained school choice," *Journal of Economic Theory*, 144, 1921–1947.

HATFIELD, J. W., F. KOJIMA, AND Y. NARITA (2016): "Improving schools through school choice: A market design approach," *Journal of Economic Theory*, 166, 186–211.

HEO, E. J. (2017): "Preference profiles for efficiency, fairness, and consistency in school choice problems," *mimeo*.

HYLLAND, A. AND R. ZECKHAUSER (1979): "The efficient allocation of individuals to positions," *Journal of Political Economy*, 91, 293–314.

JARAMILLO, P. AND V. MANJUNATH (2012): "The difference indifference makes in strategy-proof allocation of objects," *Journal of Economic Theory*, 147, 1913–1946.

KESTEN, O. (2004): "Student placement to public schools in the US: Two new solutions," *mimeo*.

——— (2006): "On two competing mechanisms for priority-based allocation problems," *Journal of Economic Theory*, 127, 155–171.

——— (2012): "On two kinds of manipulation for school choice problems," *Economic Theory*, 51, 677–693.

KLOOSTERMAN, A. AND P. TROYAN (2016): "Efficient and essentially stable assignments," *mimeo*.

KOJIMA, F. (2011): "Robust stability in matching markets," *Theoretical Economics*, 6, 257–267.

——— (2013): "Efficient resource allocation under multi-unit demand," *Games and Economic Behavior*, 82, 1–14.

KUMANO, T. (2013): "Strategy-proofness and stability of the Boston mechanism: An almost impossibility result," *Journal of Public Economics*, 105, 23–29.

MORRILL, T. (2015): "Making just school assignments," *Games and Economic Behavior*, 92, 18–27.

PÁPAI, S. (2000): "Strategyproof assignment by hierarchical exchange," *Econometrica*, 68, 1403–1433.

PYCIA, M. AND M. U. ÜNVER (2017): "Incentive compatible allocation and exchange of discrete resources," *Theoretical Economics*, 6, 287–329.

ROTH, A. E. (1982): "The economics of matching: Stability and incentives," *Mathematics of Operations Research*, 7, 617–628.

ROTH, A. E. AND M. SOTOMAYOR (1990): "Two-sided matching: A study in game-theoretic modeling and analysis," *Econometric Society Monographs. Cambridge University Press, Cambridge*.

SHAPLEY, L. AND H. SCARF (1974): "On cores and indivisibility," *Journal of Mathematical Economics*, 1, 23–37.

SÖNMEZ, T. AND M. U. ÜNVER (2005): "House allocation with existing tenants: an equivalence," *Games and Economic Behavior*, 52, 153–185.

SVENSSON, L.-G. (1994): "Queue allocation of indivisible goods," *Social Choice and Welfare*, 11, 323–330.

——— (1999): "Strategy-proof allocation of indivisible goods," *Social Choice and Welfare*, 16, 557–567.

TANG, Q. AND Y. ZHANG (2017): "Weak stability and Pareto efficiency in school choice," *mimeo*.