On the efficiency and fairness of deferred acceptance with single tie-breaking*

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Abstract

As a random allocation rule for indivisible object allocation problems under weak priorities, deferred acceptance with single tie-breaking (DA-STB) is not ex-post constrained efficient. We first observe that it also fails to satisfy a natural fairness notion, symmetry at the top, which requires that two agents be assigned their common top choice with equal probability if they have equal claim to it. Then it is shown that DA-STB is ex-post constrained efficient, if and only if it is symmetric at the top, if and only if the priority structure satisfies a certain acyclic condition. We further characterize the priority structures under which DA-STB is ex-post fully efficient. Based on the characterized priority domains, and using a mild fairness notion called local envy-freeness, some new theoretical support is provided for this widely used rule in practice. Among strategy-proof, symmetric and locally envy-free rules, there exists an ex-post constrained efficient (ex-post stable and efficient) one if and only if DA-STB is ex-post constrained efficient (ex-post stable and efficient). Among strategy-proof, ex-post stable and locally envy-free rules, there exists one that is symmetric at the top if and only if DA-STB is symmetric at the top.

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1 Introduction

We consider the priority-augmented indivisible object allocation problem, in which agents have strict preferences over objects, and the objects have priority rankings over the agents. There might be multiple copies of each object, which are to be allocated without monetary transfers. Such a problem is closely related to the two-sided matching problem (Gale and Shapley, 1962), and *school choice* (Abdulkadiroğlu and Sönmez, 2003) is one of its most important applications in practice. The stability concept from the two-sided matching theory is mainly interpreted as a fairness notion in this context: an allocation is stable if it is individually rational, nonwasteful, and there is no situation in which one agent envies another's assignment for which the first agent has a strictly higher priority. In designing a satisfactory allocation rule, fairness, efficiency and incentive compatibility are the main desiderata. However, they cannot be satisfied simultaneously, since stability is generally incompatible with efficiency (Roth, 1982, Abdulkadiroğlu and Sönmez, 2003). When the priorities are strict, the deferred acceptance (DA) algorithm from Gale and Shapley (1962) is often considered as the "best rule", which receives some strong theoretical support. First, it is the only stable and strategy-proof rule (Alcalde and Barberà, 1994).[1] Second, it selects the unique stable allocation that Pareto dominates every other stable allocation. Hence, it is stability-constrained efficient. Full efficiency can be achieved in a certain priority domain: Ergin (2002) shows that *acyclicity* on the priority structure is necessary and sufficient for DA to be an efficient rule.[2]

However, ties in priorities are common in practice. For instance, in Boston school choice programs, students are prioritized based on only two criteria (the sibling and walk zone criteria), and hence large indifference classes exist. In this paper, we focus on the general weak priority setting. In this case, random allocations are necessary to

[1]This result has been recently strengthened by Alva and Manjunath (2019): DA is the only strategy-proof and *stable-dominating* rule.

[2]It follows that acyclicity is also necessary and sufficient for the existence of a stable and efficient rule.

restore fairness with respect to the ties in priorities. A standard fairness notion in this regard is symmetry, or equal treatment of equals.[3]

In the presence of ties, there is no longer a "best rule". A natural solution is to first break the ties randomly, then apply DA. In fact, DA with single tie-breaking (DA-STB) is widely used in practice (Abdulkadiroğlu et al., 2009), in which an ordering of the agents is drawn from the uniform distribution to break the ties at all objects. While DA-STB is strategy-proof, ex-post stable and symmetric, there is a large class of random allocation rules that satisfy these properties, and it is not clear whether DA-STB has a special role among this class. Moreover, tie-breaking leads to efficiency loss and DA-STB is not constrained efficient ex-post (Erdil and Ergin, 2008). In this paper we also discover a fairness issue related to tie-breaking. One possible criterion for welfare evaluation is the agents' probabilities of obtaining their top choices, which might be particularly important in applications where the number of objects is limited but each object has a large capacity (such as in school choice). DA-STB is not *symmetric at the top*: two agents with the same top choice and the same priority for this object may not receive it with equal probability. We also define *strong symmetry*, which requires that any two agents with the same preferences and priorities for their common top k choices have the same probability of getting each of these k objects.

We first characterize the priority structures in which several desirable efficiency and fairness axioms are satisfied by DA-STB. We show that it is ex-post constrained efficient, if and only if it is symmetric at the top (strongly symmetric), if and only if the priority structure is *T-acyclic*. T-acyclicity rules out two types of cycles in the priorities that involve ties. Ehlers and Erdil (2010) show that every constrained efficient deterministic allocation is efficient if and only if the priority structure is *strongly acyclic*. We define *strong T-acyclicity* as the combination of T-acyclicity and strong acyclicity. Then it is shown that DA-STB is ex-post efficient if and only if the priority structure is strongly T-acyclic.

In light of these characterized priority domains, a natural question to ask is that, given a priority structure under which DA-STB can not deliver the desirable efficiency or fairness properties, whether some other strategy-proof, ex-post stable and symmetric rule can. The answer turns out to be no, if a rule is further required to satisfy a new

[3]For our purpose, we define a slightly stronger version of symmetry than the standard one. It requires that two agents with the same preferences and the same priorities for their common set of acceptable objects receive the same lottery.

and mild fairness axiom regarding equal priorities. Notice that symmetry is in fact a weak requirement. While it is more appropriate in *house allocation* (Hylland and Zeckhauser, 1979),[4] in our context the agents can differ not only in preferences but also in their priority rankings. We complement symmetry with *local envy-freeness*. It rules out situations in which two agents ranked equally at some object, the first agent receives this object with probability one, while the second agent, who desires this object, receives it with probability zero. Local envy-freeness can be considered as a very weak version of the original envy-freeness concept in house allocation, tailored to the setting with priorities, and it is satisfied by a large class of rules including DA-STB. We show that if the priority structure is not (strongly) T-acyclic, then there does not exist a strategy-proof, ex-post constrained efficient (ex-post stable and efficient), symmetric and locally envy-free rule, or a strategy-proof, ex-post stable, symmetric at the top and locally envy-free rule.

These impossibility results highlight the important role of DA-STB: if it fails to satisfy ex-post (constrained) efficiency, symmetry at the top or strong symmetry, then any other "reasonable" rule fails too. As far as we know, despite its popularity in practical applications, there had been almost no theoretical support for DA-STB except the well-known fact that it is strategy-proof, ex-post stable and symmetric. Abdulkadiroğlu et al. (2009) provide some support for DA with a certain fixed single tie-breaking, which is viewed as a deterministic allocation rule. They show that every constrained efficient deterministic allocation can be selected by DA with some fixed single tie-breaking. But this does not imply that DA with some fixed single tie-breaking is a constrained efficient rule.[5] It is also shown that DA with a fixed single tie-breaking cannot be Pareto dominated by a strategy-proof deterministic rule. But according to Alva and Manjunath (2019), any strategy-proof, individually rational and nonwasteful deterministic rule cannot be Pareto dominated by a strategy-proof deterministic rule. On the other hand, regarding fairness, DA-STB satisfies symmetry at the top or strong symmetry in a weakly larger priority domain than any other strategy-proof, ex-post stable and locally envy-free rule. In particular, DA with multiple tie-breaking (DA-MTB), in which for each object an ordering of the agents is drawn independently to break the ties, is symmetric at the top or strongly symmetric for a proper subset of T-acyclic priority structures. This is in contrast

[4] In house allocation, priorities are absent, or equivalently, each object (house) ranks all the agents equally.

[5] In fact, a strategy-proof and constrained efficient rule does not exist for all priority structures.

to a common perception that DA-MTB is fairer. In fact, in the design of NYC high school match DA-MTB was initially preferred to DA-STB by policy makers for fairness reasons (Abdulkadiroğlu et al., 2009).[6]

Following Ergin (2002), there is a growing literature on the characterization of priority structures in which some existing rule satisfies various desirable properties, or a certain desirable rule exists. See, for example, Kesten (2006), Ehlers (2007), Haeringer and Klijn (2009), Ehlers and Erdil (2010), Kojima (2011), Kesten (2012), Kojima (2013), Kumano (2013), Chen (2014), Tomoeda (2018), Han (2018b), Ehlers and Westkamp (2018) and Ishida (2019). All of these studies focus on deterministic rules. Our results also provide a characterization of the priority structures under which DA with any fixed single tie-breaking is an (constrained) efficient deterministic rule. In the special case of unit capacity, the weak priority structures under which a (group) strategy-proof, stable and efficient deterministic rule exists, and the ones under which a strategy-proof constrained efficient deterministic rule exists, are characterized by Han (2018b) and Ehlers and Westkamp (2018) respectively. Generalizations of the results in these two papers to the many-to-one setting remain open questions. Our results provide some partial answers, by restricting attention to random allocation rules that satisfy the fairness properties regarding ties.

In the random assignment literature, many impossibility results are established to show that fairness, efficiency and incentive compatibility can not be satisfied simultaneously. An impossibility result can emerge in the priority-free setting when group strategy-proofness is required (Bade, 2016, Zhang, 2019), or an efficiency notion defined from the ex-ante perspective is required (Zhou, 1990, Bogomolnaia and Moulin, 2001, Katta and Sethuraman, 2006, Kasajima, 2013, Martini, 2016, Chang and Chun, 2017, Nesterov, 2017). In this paper, we only consider efficiency and stability ex-post. For the allocation problem under weak priorities, Kesten and Ünver (2015), Afacan (2018), and Han (2018a) propose various new fairness properties, and construct rules to select a random allocation that is Pareto undominated within the corresponding fair allocation set. Such kinds of constrained efficiency cannot be achieved by DA-STB, or generally any strategy-proof rule.

[6]For a more thorough comparison of single tie-breaking and multiple tie-breaking, see Abdulkadiroğlu et al. (2009), Arnosti (2015), De Haan et al. (2015), Ashlagi and Nikzad (2016) and Ashlagi et al. (2019).

2 Preliminaries

Let N be a finite set of agents and A a finite set of objects. For each object $a \in A$, there are $q_a \geq 1$ copies available, and a has a complete and transitive **priority ordering** $\succeq_a$ on N, with $\succ_a$ and $\sim_a$ denoting its asymmetric and symmetric components, respectively. Given $i \in N$, let $U(\succeq_a, i) = \{j \in N : j \succeq_a i\}$ and $SU(\succeq_a, i) = \{j \in N : j \succ_a i\}$. A **priority structure** $\succeq = (\succeq_a)_{a \in A}$ is a profile of priority orderings. Let $\emptyset$ denote the null object, or the outside option, with a capacity of $q_\emptyset = +\infty$, and $\bar{A} = A \cup \{\emptyset\}$. Each agent $i \in N$ has a complete, transitive and antisymmetric **preference relation** R_i on $\bar{A}$, with P_i denoting its asymmetric component. An object $a \in A$ is **acceptable** to i if $aR_i\emptyset$. Given $a \in \bar{A}$, let $U(R_i, a) = \left\{b \in \bar{A} : bR_i a\right\}$ and $SU(R_i, a) = \left\{b \in \bar{A} : bP_i a\right\}$. A **preference profile** $R = (R_i)_{i \in N}$ is a list of individual preferences. We fix N, A and $(q_a)_{a \in A}$ in the rest of the paper. Then a **priority-augmented allocation problem**, or simply a **problem**, is represented by a pair $(\succeq, R)$.

A **random allocation**, or simply an **allocation**, is a $|N| \times |\bar{A}|$ matrix M with $M_{ia} \geq 0, \sum_{b \in \bar{A}} M_{ib} = 1$ and $\sum_{j \in N} M_{ja} \leq q_a$ for all $i \in N$ and $a \in \bar{A}$, where M_{ia} represents the probability that i is assigned a. Let $M_i = (M_{ia})_{a \in \bar{A}}$ denote the lottery obtained by i under the allocation M. An allocation M is **deterministic** if $M_{ia} \in \{0, 1\}$ for all $i \in N$ and $a \in \bar{A}$. For ease of exposition, we also use a function $\mu : N \to \bar{A}$, where $|\mu^{-1}(a)| \leq q_a$ for all $a \in \bar{A}$, to represent a deterministic allocation. Let $\mathscr{D}$ be the set of all such deterministic allocations. By a generalized version of the Birkhoff-von Neumann theorem (Birkhoff, 1946, von Neumann, 1953, Kojima and Manea, 2010), every random allocation can be represented as a lottery over deterministic allocations.[7] That is, for any allocation M, there exists a lottery λ over $\mathscr{D}$ such that $M_{ia} = \sum_{\mu \in \mathscr{D} : \mu(i) = a} \lambda(\mu)$ for all $i \in N$ and $a \in \bar{A}$, where $\lambda(\mu) \geq 0$ denotes the probability of each $\mu \in \mathscr{D}$ and $\sum_{\mu \in \mathscr{D}} \lambda(\mu) = 1$. In this case, we also say that the lottery λ **induces** M. Denote the support of a lottery λ as $\mathscr{S}(\lambda) = \{\mu \in \mathscr{D} : \lambda(\mu) > 0\}$.

Given $(\succeq, R)$, $\mu \in \mathscr{D}$ **Pareto dominates** $\nu \in \mathscr{D}$ if $\mu(i) R_i \nu(i)$ for all $i \in N$ and $\mu(j) P_j \nu(j)$ for some $j \in N$. μ is **efficient** if it cannot be Pareto dominated by any deterministic allocation. μ is **stable** if it satisfies the following conditions: (1) **individual rationality**, $\mu(i) R_i \emptyset$, $\forall i \in N$;[8] (2) **nonwastefulness**, $|\mu^{-1}(a)| < q_a$ implies $\mu(i) R_i a$,

[7] There could be multiple lottery representations of a random allocation. Also, see Budish et al. (2013) for a maximal generalization of this theorem to a broader class of allocation problems.

[8] We also say that a random allocation M is individually rational, if for all $i \in N$ and $a \in A$, $M_{ia} > 0$ implies $aR_i\emptyset$.

$\forall a \in A, i \in N$; (3) **respecting priorities**, $\mu(j) P_i \mu(i)$ implies $j \succeq_{\mu(j)} i$, $\forall i, j \in N$. Then, a random allocation M is **ex-post efficient** (resp., **ex-post stable**) if some λ induces M and each $\mu \in \mathscr{S}(\lambda)$ is efficient (resp., stable). Similarly, M is **ex-post stable and efficient** if some λ induces M and each $\mu \in \mathscr{S}(\lambda)$ is both stable and efficient. If M is ex-post stable and efficient, then clearly it is both ex-post stable and ex-post efficient. However, the converse might not be true.

Example 1. Suppose that $N = \{1, 2, 3, 4, 5\}$, $A = \{a, b, c, d, e\}$ and all the objects have unit-capacity. The priority structure is as follows: (1) $i \sim_o j$ for all $i, j \in N \setminus \{5\}$ and $o \in A$; (2) $i \succ_o 5$ for all $i \in N \setminus \{5\}$ and $o \in A$. Consider the following preference profile and allocation M, where an underlined object is assigned to the corresponding agent with probability 0.5.

$1 : \underline{a}, \underline{b}, \emptyset$
$2 : \underline{a}, e, \underline{b}, \emptyset$
$3 : \underline{d}, \underline{c}, \emptyset$
$4 : b, \underline{d}, \underline{e}, \emptyset$
$5 : \underline{e}, \underline{c}, \emptyset$

For simplicity, we use $\mu^1 = (a, b, c, d, e)$ to denote the deterministic allocation μ^1 in which $\mu^1(1) = a, \mu^1(2) = b, \mu^1(3) = c, \mu^1(4) = d$ and $\mu^1(5) = e$. Let $\mu^2 = (b, a, d, e, c)$, $\mu^3 = (a, b, d, e, c)$ and $\mu^4 = (b, a, c, d, e)$. Then both the lottery λ, in which $\lambda(\mu^1) = \lambda(\mu^2) = \frac{1}{2}$, and the lottery λ', in which $\lambda'(\mu^3) = \lambda'(\mu^4) = \frac{1}{2}$, induce M. It can be easily verified that μ^1 and μ^2 are efficient, and μ^3 and μ^4 are stable. Therefore, M is ex-post stable and ex-post efficient. However, it is not ex-post stable and efficient. To see this, let λ'' be any lottery that induces M. There exists $\mu \in \mathscr{S}(\lambda'')$ such that $\mu(2) = b$. Suppose that μ is efficient. Then $\mu(4) = d$. Since $M_{3d} = M_{3c} = M_{5c} = M_{5e} = \frac{1}{2}$, we have $\mu(3) = c$ and $\mu(5) = e$. But $e P_2 b$ and $2 \succ_e 5$, so μ is not stable. Therefore, some deterministic allocation in the support of λ'' cannot be both stable and efficient.

Given a priority structure $\succeq$, a **rule** is a function that maps each preference profile to a random allocation. A rule f is said to satisfy a certain property defined above if $f(R)$ satisfies this property for all R. f is **strategy-proof** if for each agent truth-telling yields a lottery that first-order stochastically dominates the lottery obtained from reporting any other preferences: for any $i \in N$, $a \in \bar{A}$, R and R'_i, $\sum_{b \in U(R_i, a)} f_{ib}(R) \geq \sum_{b \in U(R_i, a)} f_{ib}(R'_i, R_{-i})$.

The purpose of using random allocations is to treat agents with equal priorities in a fair way. A standard fairness notion in this regard is *symmetry*, or *equal treatment of equals*, which requires that any two agents with the same preferences and the same priority at each object should receive the same lottery. For our purpose, we define a slightly stronger version of this notion. Given $(\succeq, R)$, an allocation M is **symmetric** if for any $i, j \in N$ such that $U(R_i, \emptyset) = U(R_j, \emptyset)$, $R_i|_{U(R_i,\emptyset)} = R_j|_{U(R_i,\emptyset)}$, and $i \sim_a j$ for all $a \in U(R_i, \emptyset)$, we have $M_i = M_j$.[9] So two agents are considered to be "equals" if they only have different preferences and/or priorities regarding their common set of unacceptable objects. This strengthening of the original symmetry notion can be justified if we only focus on individually rational rules and require that the allocation recommended by a rule does not vary with preferences or priorities regarding unacceptable objects.

3 Deferred acceptance with single tie-breaking

The rule of central interest in this paper is deferred acceptance with single tie-breaking. To introduce this rule, we first consider the special case of strict priorities. That is, consider some $\succeq$ such that $\succeq_a$ is antisymmetric for all $a \in A$. In this case, we can restrict attention to deterministic allocations, since no randomization is needed to deal with the fairness issue regarding ties. Given any R, there exists an *agent-optimal stable* deterministic allocation, i.e., a stable deterministic allocation that Pareto dominates any other stable deterministic allocation. This allocation is given by the following **deferred acceptance algorithm** (DA) from Gale and Shapley (1962).

Step 1. Each agent applies to her favorite acceptable object. Each object a places the applicants with the highest priorities up to its quota q_a on its waiting list, and rejects all the other applicants.

Step *k*. Each agent who was rejected in the step $k-1$ applies to her next best acceptable object. Each object a chooses among the new applicants and the applicants on its waiting list, places the ones with the highest priorities up to its quota q_a on its waiting list, and rejects all others.

The process terminates when every agent (who has at least one acceptable object) is either rejected by all of her acceptable objects or on the waiting list of some object. Then the copies of each object are assigned to the agents on its waiting list.

[9] $R_i|_{U(R_i,\emptyset)}$ is the restriction of R_i to $U(R_i, \emptyset)$.

DA is strategy-proof (Dubins and Freedman, 1981, Roth, 1982). However, it may not be efficient. Ergin (2002) characterizes the priority structures under which DA is an efficient rule. A **cycle** consists of distinct $i, j, k \in N$ and $a, b \in A$ such that the following conditions are satisfied: (Cycle condition) $i \succ_a j \succ_a k \succ_b i$; (Scarcity condition) there exist (possibly empty) disjoint sets $N_a, N_b \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq SU(\succeq_a, j)$, $N_b \subseteq SU(\succeq_b, i)$, $|N_a| = q_a - 1$ and $|N_b| = q_b - 1$. Then the priority structure $\succeq$ is **acyclic** if there does not exist any cycle. Ergin (2002) shows that DA is efficient if and only if $\succeq$ is acyclic.[10]

Next, consider a weak priority structure $\succeq$. We will also fix this priority structure in the rest of the paper. A natural and common solution in this case is to first break all the ties randomly, then apply DA to the resulting strict priority structure, which gives rise to a random allocation rule. To achieve fairness, an ordering of the agents can be drawn from the uniform distribution to break the ties at all the objects (*single tie-breaking*). Alternatively, for each object an ordering can be drawn independently to break the ties at this object (*multiple tie-breaking*). Pathak and Sethuraman (2011) introduced a more general method of tie-breaking: we can partition all the objects into a collection of subsets, and conduct a lottery draw independently for each subset.

Formally, an ordering of the agents is a one-to-one function $\sigma : N \to \{1, ..., |N|\}$. Denote the set of all such orderings as $\mathcal{O}$, and $A = \{a_1, ..., a_{|A|}\}$. A list of orderings $\varsigma = (\varsigma_{a_1}, ..., \varsigma_{a_{|A|}}) \in \mathcal{O}^{|A|}$ transforms $\succeq$ into a strict priority structure $\succeq^{\varsigma}$: for all $i, j \in N$ and $a \in A$, $i \succ_a^{\varsigma} j$ if $i \succ_a j$, or, $i \sim_a j$ and $\varsigma_a(i) < \varsigma_a(j)$. Let $f^{DA}(\varsigma, R)$ denote the deterministic allocation obtained from applying DA to $\succeq^{\varsigma}$ and R. Given R, and a partition $\mathcal{P} = \{A_1, ..., A_k\}$ of A, **DA with $\mathcal{P}$-partitioned tie-breaking** selects the allocation $f^{DA\text{-}\mathcal{P}}(R)$, where for all $i \in N$ and $a \in \bar{A}$,

$$f_{ia}^{DA\text{-}\mathcal{P}}(R) = \frac{1}{(|N|!)^k} \left| \left\{ \varsigma : \varsigma_b = \varsigma_c \text{ if } b, c \in A_l \text{ for some } l = 1, \ldots, k, f_i^{DA}(\varsigma, R) = a \right\} \right|$$

If $|\mathcal{P}| = |A|$, then $f^{DA\text{-}\mathcal{P}} = f^{DA\text{-}MTB}$ is **DA with multiple tie-breaking** (DA-MTB). If $\mathcal{P} = \{A\}$, then $f^{DA\text{-}\mathcal{P}} = f^{DA\text{-}STB}$ is **DA with single tie-breaking** (DA-STB). We will mostly focus on DA-STB. So, for simplicity, denote $\tilde{\mathcal{O}}^{|A|} = \left\{ \varsigma \in \mathcal{O}^{|A|} : \varsigma_a = \varsigma_b \text{ for all } a, b \in A \right\}$.

In the special case of strict priorities, the use of DA is strongly justified: it is the only stable and strategy-proof rule (Alcalde and Barberà, 1994); it is agent-optimal stable, hence stability-constrained efficient. In contrast, under weak priorities, there is

[10] He further shows that DA is consistent, or group strategy-proof, if and only if $\succeq$ is acyclic.

no such support for DA-STB. First, there is a large class of symmetric, ex-post stable and strategy-proof random allocation rules. For instance, it is easy to see that DA with any partitioned tie-breaking belongs to this class. Second, as shown in Erdil and Ergin (2008), tie-breaking could induce further efficiency loss: for some $\varsigma \in \tilde{\mathscr{O}}^{|A|}$ and R, $f^{DA}(\varsigma, R)$ may be Pareto dominated by another stable deterministic allocation. Therefore, DA-STB may not be even stability-constrained efficient ex-post. Formally, given R, $\mu \in \mathscr{D}$ is **constrained efficient** if it is stable and cannot be Pareto dominated by any stable deterministic allocation. A random allocation M is **ex-post constrained efficient** if some lottery λ induces M and each $\mu \in \mathscr{S}(\lambda)$ is constrained efficient.

Example 2. Suppose that $N = \{1, 2, 3\}$, $A = \{a, b\}$ and $q_a = q_b = 1$. The priority structure $\succeq$ and preference profile R are given as follows.

$$\begin{array}{llll} a: & 1 \succ_a 2 \sim_a 3 & 1: & b, a, \emptyset \\ b: & 1 \sim_b 2 \sim_b 3 & 2: & a, \emptyset \\ & & 3: & a, b, \emptyset \end{array}$$

If $(\sigma(1), \sigma(2), \sigma(3)) = (3, 1, 2)$ and $\varsigma_a = \varsigma_b = \sigma$, then $f_1^{DA}(\varsigma, R) = a$. Hence $f_{1a}^{DA\text{-}STB}(R) > 0$. Let λ be any lottery that induces $f^{DA\text{-}STB}(R)$, then there exists $\mu \in \mathscr{S}(\lambda)$ such that $\mu(1) = a$. If μ is nonwasteful and individually rational, then $\mu(2) = \emptyset$ and $\mu(3) = b$. μ is Pareto dominated by the stable deterministic allocation ν with $(\nu(1), \nu(2), \nu(3)) = (b, \emptyset, a)$. Therefore, DA-STB is not ex-post constrained efficient.

In this example, when the ties are broken using ς, a rejection cycle emerges in DA: in the first step, agent 2 is tentatively accepted and agent 3 is rejected by a; in the second step agent 3 displaces agent 1 at b; in the third step agent 1 displaces agent 2 at a. This cycle leads to efficiency loss, and a fairness issue as well. When the tie at object a is broken in favor of agent 2, she may not get a in the end. But note that agent 3 always gets a if the tie is broken in favor of her. Therefore, although these two agents have equal claim to their common top choice, they are not assigned this object with equal probability under DA-STB. In fact, $f_{2a}^{DA\text{-}STB}(R) = \frac{1}{3}$ and $f_{3a}^{DA\text{-}STB}(R) = \frac{1}{2}$.

Definition 1 Given R, an allocation M is **symmetric at the top** if for any $i, j \in N$ and $a \in A$ such that $i \sim_a j$ and $U(R_i, a) = U(R_j, a) = \{a\}$, we have $M_{ia} = M_{ja}$.[11]

[11] In a house allocation problem, symmetry at the top is equivalent to *equal-top fairness* defined by Zhang (2019).

We can also define a stronger fairness notion in this regard: if two agents have the same first k choices, and they have the same preferences and priorities for these objects, then they are assigned each of these k objects with equal probability.

Definition 2 Given R, an allocation M is **strongly symmetric** if for any $i, j \in N$ and $a \in A$ such that $\emptyset \notin U(R_i, a) = U(R_j, a)$, $R_i|_{U(R_i,a)} = R_j|_{U(R_i,a)}$ and $i \sim_b j$ for all $b \in U(R_i, a)$, we have $M_{ib} = M_{jb}$ for all $b \in U(R_i, a)$.

Strong symmetry implies symmetry at the top. It also implies symmetry for individually rational allocations. But there is no logical relation between symmetry at the top and symmetry. Finally, using the problem constructed in Example 2, it is easy to see that DA with any partitioned tie-breaking is generally not ex-post constrained efficient or symmetric at the top.

4 Results

We are first interested in characterizing the priority structures under which DA-STB satisfies good efficiency or fairness properties. As suggested by Example 2, for DA-STB ex-post constrained efficiency and symmetry at the top fail for similar reasons. As will be seen in Theorem 1, it turns out that the priority domain in which DA-STB is ex-post constrained efficient coincides with the one in which it is symmetric at the top or strongly symmetric. This domain is obtained by ruling out two types of cycles involving ties.

Definition 3 A **type-I cycle** consists of $n \geq 3$ distinct agents $i_1, ..., i_n \in N$ and $n-1$ distinct objects $a_1, ..., a_{n-1} \in A$ such that the following conditions are satisfied: (Cycle) $i_1 \succ_{a_1} i_2 \sim_{a_1} i_3$, and $i_{k+1} \succeq_{a_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$; (Scarcity) there exist $n-1$ (possibly empty) mutually disjoint sets $N_1, ..., N_{n-1} \subseteq N \setminus \{i_1, ..., i_n\}$ such that $N_k \subseteq U(\succeq_{a_k}, i_{k+2})$ and $|N_k| = q_{a_k} - 1$ for each k with $1 \leq k \leq n-1$.

A **type-II cycle** consists of distinct $i, j, k \in N$ and $a, b \in A$ such that the following conditions are satisfied: (Cycle) $i \sim_a j \sim_a k \succ_b i$; (Scarcity) there exist (possibly empty) disjoint sets $N_a, N_b \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq U(\succeq_a, i)$, $N_b \subseteq U(\succeq_b, i)$, $|N_a| = q_a - 1$ and $|N_b| = q_b - 1$.

The priority structure $\succeq$ is **T-acyclic** if there does not exist any type-I or type-II cycle.

It is also interesting to know that, when T-acyclicity is not satisfied, whether there exists a satisfactory rule that performs better than DA-STB. We will show that generally such a rule does not exist, if it is required to satisfy an additional fairness property that deals with ties. Notice that symmetry is indeed a weak axiom in our context, as it is silent about two "unequal" agents' equal claim to some object. For instance, if every two agents have different priority rankings at only one object (that is acceptable to at least one of them), then symmetry does not impose any restriction and every deterministic allocation is symmetric. While symmetry focuses on the whole probability distributions for equal agents, the following axiom imposes restrictions on the allocation of each object among equally ranked agents.

Definition 4 Given R, an allocation M is **locally envy-free** if there do not exist $i, j \in N$ and $a \in A$ such that $i \sim_a j$, $M_{ia} = 1$, $M_{ja} = 0$ and $\sum_{b \in U(R_j,a)} M_{jb} < 1$.

In house allocation, envy-freeness requires that for each agent, her lottery first-order stochastically dominates any other agent's lottery. This is clearly not appropriate in our model: if M_j does not first-order stochastically dominate M_i, then j's (potential) envy may not be "justified" as i may have higher priorities for certain objects. However, if $j \succeq_a i$ for every $a \in A$ with $M_{ia} > 0$, then j's envy is justified and should be eliminated. Local envy-freeness is then defined by ruling out one of the most obvious types of such justified envy. In many cases it also requires a rule to be minimally random: if two agents are ranked equally at an object that they both desire, and there is only one copy of this object to be allocated to them, then the rule cannot simply assign this copy to one agent in a deterministic way. Local envy-freeness is satisfied by many common random allocation rules.

Lemma 1 *$f^{DA\text{-}\mathscr{P}}$ is locally envy-free for any partition $\mathscr{P}$ of A.*

In house allocation, the two most studied rules, *random serial dictatorship* (Abdulkadiroğlu and Sönmez, 1998) and *probabilistic serial rule* (Bogomolnaia and Moulin, 2001) are locally envy-free.[12] Moreover, in allocation problems with weak priorities, several recent studies propose random allocation rules that satisfy new fairness axioms regarding ties in priorities. Local envy-freeness is implied by *no ex-ante discrimination*

[12] In fact, the probabilistic serial rule satisfies the stronger notion of envy-freeness, while random serial dictatorship, which is a special case of DA-STB, is not envy-free.

from Kesten and Ünver (2015) as well as *ordinal fairness* from Han (2018a).[13] Although there is no logical relation between local envy-freeness and *claimwise stability* from Afacan (2018), the central claimwise stable rule constructed by him, *constrained probabilistic serial mechanism*, is locally envy-free.[14]

We are ready to present the first main result.

Theorem 1 *The following statements are equivalent:*

(i) *DA-STB is ex-post constrained efficient.*

(ii) *DA-STB is strongly symmetric.*

(iii) *There exists a strategy-proof, ex-post constrained efficient, symmetric and locally envy-free rule.*

(iv) *There exists a strategy-proof, ex-post stable, symmetric at the top and locally envy-free rule.*

(v) $\succeq$ *is T-acyclic.*

It can be easily seen that symmetry at the top and strong symmetry are interchangeable in the above statements. To find the priority structures under which DA-STB is ex-post efficient, we only need to complement T-acyclicity with the *strong acyclicity* condition from Ehlers and Erdil (2010). A **weak cycle** consists of distinct $i, j, k \in N$ and $a, b \in A$ such that the following conditions are satisfied: (Cycle) $i \succeq_a j \succ_a k \succeq_b i$; (Scarcity) there exist (possibly empty) disjoint sets $N_a, N_b \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq U(\succeq_a, j)$, $N_b \subseteq U(\succeq_b, i)$, $|N_a| = q_a - 1$ and $|N_b| = q_b - 1$. $\succeq$ is **strongly acyclic** if there does not exist any weak cycle.

Definition 5 The priority structure $\succeq$ is **strongly T-acyclic** if it is T-acyclic and strongly acyclic.

Ehlers and Erdil (2010) show that every constrained efficient deterministic allocation under any preference profile is efficient if and only if $\succeq$ is strongly acyclic. Therefore, DA-STB is ex-post stable and efficient if $\succeq$ is strongly T-acyclic. The other direction

[13] An allocation M has no ex-ante discrimination if for any $i, j \in N$ and $a \in A$ with $i \sim_a j$, $\sum_{b \in U(R_j, a)} M_{jb} < 1$ implies $M_{ja} \geq M_{ia}$. M is ordinally fair if for any $i, j \in N$ and $a \in A$ with $i \sim_a j$, $M_{ia} > 0$ implies $\sum_{b \in U(R_j, a)} M_{jb} \geq \sum_{b \in U(R_i, a)} M_{ib}$.

[14] Afacan (2018) studies a more general model in which priorities are random. An allocation M is claimwise stable if for any $i, j \in N$ and $a \in A$, $M_{ia} \leq Pr(i \succ_a j) + \sum_{b \in SU(R_j, a)} M_{jb}$, where $Pr(i \succ_a j)$ is the probability of $i \succ_a j$. Our model can be embedded into his by setting $Pr(i \succ_a j) = \frac{1}{2}$ when $i \sim_a j$.

can also be established, through a general (im)possibility result similar to those in Theorem 1.

Theorem 2 *The following statements are equivalent:*

(i) *DA-STB is ex-post stable and efficient.*

(ii) *There exists a strategy-proof, ex-post stable and efficient, symmetric and locally envy-free rule.*

(iii) $\succeq$ *is strongly T-acyclic.*

Notice that, in light of Example 1, when full efficiency is desired in addition to stability, we want the rule to satisfy ex-post stability and efficiency, rather than ex-post stability plus ex-post efficiency.[15] Three impossibility results are established in the proofs of Theorem 1 and 2. In Appendix B we show the axioms involved are generally independent, while there are some stronger impossibility results in the special one-to-one matching case.

5 Discussion

We conclude by discussing the main implications of Theorem 1 and 2.

First, the necessary and sufficient conditions for DA-STB to satisfy several desirable efficiency and fairness axioms are identified. While previous studies on characterizations of priority domains all focus on the deterministic setting, our results also have implications for deterministic allocation rules. The proof of Theorem 1 indicates that $f^{DA}(\varsigma, \cdot)$ is a constrained efficient deterministic allocation rule for all $\varsigma \in \tilde{\mathcal{O}}^{|A|}$ if and only if $\succeq$ is T-acyclic. Then it follows that $f^{DA}(\varsigma, \cdot)$ is an efficient rule for all $\varsigma \in \tilde{\mathcal{O}}^{|A|}$ if and only if $\succeq$ is strongly T-acyclic.[16] Moreover, Abdulkadiroğlu et al. (2009) shows that any constrained efficient allocation can be selected by DA with some fixed single tie-breaking. Therefore, under any preference profile, DA with fixed single tie-breaking characterizes the set of constrained efficient allocations, i.e., for any R the set of constrained efficient allocations is equal to $\{f^{DA}(\varsigma, R) : \varsigma \in \tilde{\mathcal{O}}^{|A|}\}$, if and only if $\succeq$ is T-acyclic. Similarly, under any preference profile DA with fixed single tie-breaking characterizes

[15] That being said, it can be shown that in the statement (i) of Theorem 2, ex-post stability and efficiency can be replaced by ex-post efficiency. So DA-STB is an ex-post stable and efficient rule if and only if it is an ex-post stable and ex-post efficient rule.

[16] In a related study, Ishida (2019) characterizes the priority structures under which applying DA after the ties in priorities are resolved in any way results in an efficient rule.

the set of stable and efficient allocations if and only if $\succeq$ is strongly T-acyclic.[17] Finally, for the general many-to-one allocation problem under weak priorities, it is still unknown when a strategy-proof, stable and efficient rule exists, and when a strategy-proof and constrained efficient rule exists.[18] Our results provide some partial answers by restricting attention to random allocation rules that are fair with respect to the ties in priorities.

Second, despite its popularity in market design applications, a theoretical support for the use of DA-STB, beyond the fact that it is strategy-proof, ex-post stable and symmetric, has been largely elusive, and this paper attempts to fill this gap. If we take strategy-proofness, ex-post stability, symmetry and local envy-freeness as the minimal requirements of a desirable random allocation rule, then ex-post constrained efficiency, ex-post stability and full efficiency, symmetry at the top or strong symmetry can be achieved if and only if it can be achieved by DA-STB. Therefore, a market designer who values, for instance, constrained efficiency, has a stronger reason to use DA-STB without examining the actual priority structure, since an ex-post constrained efficient rule satisfying those minimal requirements does not exist if DA-STB is not ex-post constrained efficient. Of course, DA-STB might not be the only rule that can be justified in this way. However, it can be easily shown that, compared to DA-STB, at least DA with any other partitioned tie-breaking satisfies the corresponding desirable efficiency and fairness properties in strictly smaller priority domains.

Appendix A: Proofs

Proof of Lemma 1

Assume to the contrary, there exist a partition $\mathscr{P}$ of A, R, $i, j \in N$ and $a \in A$ such that $i \sim_a j$, $f_{ia}^{DA\text{-}\mathscr{P}}(R) = 1$, $f_{ja}^{DA\text{-}\mathscr{P}}(R) = 0$ and $\sum_{b \in U(R_j,a)} f_{jb}^{DA\text{-}\mathscr{P}}(R) < 1$. Then for some $\varsigma \in \mathscr{O}^{|A|}$ with $\varsigma_b = \varsigma_c$ for all $A' \in \mathscr{P}$ and $b, c \in A'$, $f_i^{DA}(\varsigma, R) = a$ and $aP_j f_j^{DA}(\varsigma, R)$. By the stability of DA, $\varsigma_a(i) < \varsigma_a(j)$. Next, we adjust the ordering ς_a by moving i down such that i is below j. That is, consider some $\varsigma'_a \in \mathscr{O}$ such that $\varsigma'_a(j) < \varsigma'_a(i)$, and for

[17] This implies a well-known result that in a house allocation problem, serial dictatorships characterize the set of efficient allocations (Svensson, 1994).

[18] Han (2018b) and Ehlers and Westkamp (2018) answer these questions in the one-to-one setting, respectively.

all $k \in N \setminus \{i\}$ and $l \in N$, $\varsigma'_a(k) < \varsigma'_a(l)$ if $\varsigma_a(k) < \varsigma_a(l)$. Let $\varsigma' = (\varsigma'_a, \varsigma_{-a})$. We want to show that $f_i^{DA}(\varsigma', R) \neq a$. Suppose that this is not true. Then by the stability of DA, $f_j^{DA}(\varsigma', R) R_j a$. Consider any $k, l \in N$ such that $f_l^{DA}(\varsigma', R) P_k f_k^{DA}(\varsigma', R)$. Let $f_l^{DA}(\varsigma', R) = b$. By the stability of DA, $l \succ_b^{\varsigma'} k$. If $b \neq a$, then $l \succ_b^{\varsigma} k$ since $\varsigma_b = \varsigma'_b$. If $b = a$, then $k \neq i$ and the construction of ς'_a implies that we also have $l \succ_b^{\varsigma} k$. So $f^{DA}(\varsigma', R)$ respects the priorities in the problem $(\succeq^{\varsigma}, R)$. It is also individually rational and nonwasteful for this problem since it is so for $(\succeq^{\varsigma'}, R)$. Therefore, $f^{DA}(\varsigma', R)$ is stable for $(\succeq^{\varsigma}, R)$. But $f_j^{DA}(\varsigma', R) R_j a P_j f_j^{DA}(\varsigma, R)$, contradicting to the fact that DA is agent-optimal stable. Hence, $f_i^{DA}(\varsigma', R) \neq a$.

Balinski and Sönmez (1999) show that DA *respects improvements*. That is, if an agent's priority ranking at each object weakly increases, she will receive a weakly better object under DA. From $\succeq^{\varsigma}$ to $\succeq^{\varsigma'}$, agent i's ranking at each object weakly decreases, so $f_i^{DA}(\varsigma', R) \neq a$ implies $a P_i f_i^{DA}(\varsigma', R)$. Suppose that $a \in A' \in \mathscr{P}$. Construct ς'' such that for all $c \in A$, $\varsigma''_c = \varsigma'_a$ if $c \in A'$, and $\varsigma''_c = \varsigma_c$ otherwise. From $\succeq^{\varsigma'}$ to $\succeq^{\varsigma''}$, agent i's ranking at each object weakly decreases again, so $a P_i f_i^{DA}(\varsigma'', R)$. This contradicts to the initial assumption of $f_{ia}^{DA\text{-}\mathscr{P}}(R) = 1$. □

Proof of Theorem 1

We first show $(v) \Rightarrow (i)$, then $(v) \Rightarrow (ii)$. It is already known that $(i) \Rightarrow (iii)$ and $(ii) \Rightarrow (iv)$. So we finish the proof by showing that $(iii) \Rightarrow (v)$ and $(iv) \Rightarrow (v)$.

We start by proving a result regarding longer type-II cycles. A **generalized type-II cycle** consists of $n \geq 3$ distinct agents $i_1, ..., i_n \in N$ and $n-1$ distinct objects $a_1, ..., a_{n-1} \in A$ such that the following conditions are satisfied: (Cycle) $i_1 \sim_{a_1} i_2 \sim_{a_1} i_3$, $i_{k+1} \succeq_{a_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, and $i_{k+1} \succ_{a_k} i_{k+2}$ for some k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$; (Scarcity) there exist $n-1$ (possibly empty) mutually disjoint sets $N_1, ..., N_{n-1} \subseteq N \setminus \{i_1, ..., i_n\}$ such that $N_k \subseteq U(\succeq_{a_k}, i_{k+2})$ and $|N_k| = q_{a_k} - 1$ for each k with $1 \leq k \leq n-1$.

Claim 1. *If there is a generalized type-II cycle, there is a type-I cycle or a type-II cycle.*

Proof. Suppose that there is a generalized type-II cycle. Consider one of the shortest generalized type-II cycles. Assume that it consists of $n \geq 3$ distinct agents $i_1, ..., i_n$ and $n-1$ distinct objects $a_1, ..., a_{n-1}$, which satisfy the cycle and scarcity conditions in the definition above. If $n = 3$, then this is a type-II cycle. Suppose that $n \geq 4$ and consider

the following three possible cases.

Case 1. $i_1 \not\sim_{a_{n-1}} i_2$ or $i_1 \not\sim_{a_{n-1}} i_3$. The three agents i_1, i_2, i_3 and the two objects a_1, a_{n-1} constitute a shorter generalized type-II cycle. So this case is not possible.

Case 2. $i_n \succ_{a_{n-1}} i_1 \sim_{a_{n-1}} i_2 \sim_{a_{n-1}} i_3$. Since $i_n \succ_{a_{n-1}} i_1 \sim_{a_{n-1}} i_2$, $i_{k+1} \succeq_{a_k} i_{k+2}$ for $k = 1, ..., n-2$, and it is easy to check that the scarcity condition is satisfied, the agents $i_n, i_1, ..., i_{n-1}$ and the objects $a_{n-1}, a_1 ..., a_{n-2}$ constitute a type-I cycle.

Case 3. $i_n \sim_{a_{n-1}} i_1 \sim_{a_{n-1}} i_2 \sim_{a_{n-1}} i_3$. First, $i_n \sim_{a_{n-1}} i_1$ implies that for some k with $2 \leq k \leq n-2$, $i_{k+1} \succ_{a_k} i_{k+2}$. Then, we have $i_n \sim_{a_{n-1}} i_1 \sim_{a_{n-1}} i_3$, and $i_{k+1} \succeq_{a_k} i_{k+2}$ for all k with $2 \leq k \leq n-2$. The scarcity condition can also be easily verified, so a shorter generalized type-II cycle is found, which consists of $n-1$ agents $i_n, i_1, i_3, ..., i_{n-1}$ and $n-2$ objects $a_{n-1}, a_2, ..., a_{n-2}$. Hence this case is not possible. □

Proof of *(v)* ⇒ *(i)*. Suppose that there exist R and $\varsigma \in \tilde{\mathcal{O}}^{|A|}$ such that $f^{DA}(\varsigma, R) = \mu$ is not constrained efficient for the problem $(\succeq, R)$. Then, by Erdil and Ergin (2008), there exists a *stable improvement cycle*. That is, there exists a sequence of $n \geq 2$ distinct agents $(i_1, ..., i_n)$ such that for all $k \in \{1, ..., n\}$, $\mu(i_k) \in A$, $\mu(i_{k+1}) P_{i_k} \mu(i_k)$, and $i_k \succeq_{\mu(i_{k+1})} i$ for all i with $\mu(i_{k+1}) P_i \mu(i)$, where $i_{n+1} = i_1$. Without loss of generality, assume that this is one of the shortest stable improvement cycles, and among the shortest ones this cycle minimizes the sum of the involved agents' orderings. That is, in every stable improvement cycle there are at least n agents, and if $(i'_1, ..., i'_n)$ is a stable improvement cycle, then $\sum_{k=1}^{n} \sigma(i_k) \leq \sum_{k=1}^{n} \sigma(i'_k)$, where $\sigma = \varsigma_a$ for all $a \in A$. The objects involved in the cycle $(i_1, ..., i_n)$ must be distinct. To see this, consider any $k, k' \in \{1, ..., n\}$ such that $k' > k$. If $k' - k = 1$, then $\mu(i_{k'}) \neq \mu(i_k)$ since $\mu(i_{k'}) P_{i_k} \mu(i_k)$. If $k' - k > 1$, then $\mu(i_{k'}) = \mu(i_k)$ implies that $(i_k, ..., i_{k'-1})$ is a shorter stable improvement cycle.

Define μ' as follows: $\mu'(i) = \mu(i)$ if $i \notin \{i_1, ..., i_n\}$; $\mu'(i_k) = \mu(i_{k+1})$ for all $k \in \{1, ..., n\}$. Then μ' is stable for $(\succeq, R)$ and it Pareto dominates $\mu = f^{DA}(\varsigma, R)$. It follows that μ' is not stable for $(\succeq^{\varsigma}, R)$. Clearly μ' is individually rational and nonwasteful for $(\succeq^{\varsigma}, R)$. Hence, there exist $i, j \in N$ such that $j \succ^{\varsigma}_{\mu'(i)} i$ and $\mu'(i) P_j \mu'(j)$. Then $\mu'(i) P_j \mu(j)$. By the stability of μ for $(\succeq^{\varsigma}, R)$, it must be the case that $\mu(i) \neq \mu'(i)$. So $i \in \{i_1, ..., i_n\}$. Without loss of generality, let $i = i_1$, then $\mu'(i) = \mu(i_2)$. Since $\mu(i_2) P_j \mu(j)$, by the definition of stable improvement cycles, $i_1 \succeq_{\mu(i_2)} j$. Then it follows from $j \succ^{\varsigma}_{\mu(i_2)} i_1$ that $i_1 \sim_{\mu(i_2)} j$ and and $\sigma(j) < \sigma(i_1)$.

Next, we show that $\mu(j) \neq \mu(i_k)$ for any $k \in \{1, ..., n\}$. Given that $\mu(i_2) P_j \mu(j)$, $\mu(j) \neq \mu(i_2)$. Suppose $\mu(j) = \mu(i_1)$. Then $j \notin \{i_2, ..., i_n\}$ as $\mu(i_1), ..., \mu(i_n)$ are distinct. Since $i_1 \sim_{\mu(i_2)} j$ and $\mu(i_2) P_j \mu(j)$, we can replace i_1 with j and obtain another stable

improvement cycle, $(j, i_2, ..., i_n)$, with the same length. However, a contradiction is reached since the sum of the involved agents' orderings is smaller in the new stable improvement cycle: $\sum_{k=1}^{n} \sigma(i_k) > \sigma(j) + \sum_{k=2}^{n} \sigma(i_k)$. Finally, suppose that $n \geq 3$ and $\mu(j) = \mu(i_k)$ for some k with $3 \leq k \leq n$. In this case, $j \notin \{i_2, ..., i_{k-1}\}$, and $(j, i_2, ..., i_{k-1})$ is a stable improvement cycle with less than n agents, contradiction.

Since μ is stable for $(\succeq, R)$ and $\mu(i_{k+1}) P_{i_k} \mu(i_k)$ for all $k \in \{1, ..., n\}$, $i_{k+1} \succeq_{\mu(i_{k+1})} i_k$ for all k. For each $k \in \{1, ..., n\}$, let $N_k = \{l \in N : \mu(l) = \mu(i_k), l \neq i_k\}$. Then by the stability of μ again, $N_{k+1} \subseteq U(\succeq_{\mu(i_{k+1})}, i_k)$ and $|N_{k+1}| = q_{\mu(i_{k+1})} - 1$ for all $k \in \{1, ..., n\}$, where $N_{n+1} = N_1$. It is obvious that $N_1, ..., N_n$ are mutually disjoint, and $N_k \subseteq N \setminus \{i_1, ..., i_n\}$ for each $k \in \{1, ..., n\}$. Given that $\mu(j) \neq \mu(i_k)$ for any k, $j \notin \{i_1, ..., i_n\}$ and $j \notin N_k$ for any k.

In sum, we have $n+1$ distinct agents $j, i_1, ..., i_n$ and n distinct objects $\mu(i_1), ..., \mu(i_n)$ such that: (1) $i_2 \succeq_{\mu(i_2)} j \sim_{\mu(i_2)} i_1$, $i_{k+1} \succeq_{\mu(i_{k+1})} i_k$ for $k = n, ..., 2$; (2) there exist n (possibly empty) mutually disjoint sets $N_1, ..., N_n \subseteq N \setminus \{j, i_1, ..., i_n\}$ such that $N_{k+1} \subseteq U(\succeq_{\mu(i_{k+1})}, i_k)$ and $|N_{k+1}| = q_{\mu(i_{k+1})} - 1$ for all $k \in \{1, ..., n\}$. This is a type-I cycle if $i_2 \succ_{\mu(i_2)} i_1$. If $i_2 \sim_{\mu(i_2)} i_1$, then $i_{k+1} \succ_{\mu(i_{k+1})} i_k$ for some $k \in \{2, ..., n\}$, because otherwise the stability of μ for $(\succeq^{\varsigma}, R)$ implies $\sigma(i_1) > \sigma(i_2) > ... > \sigma(i_n) > \sigma(i_1)$. Hence, in the case of $i_2 \sim_{\mu(i_2)} i_1$ we obtain a generalized type-II cycle. By Claim 1, there exists a type-I cycle or a type-II cycle.

Therefore, if $\succeq$ is T-acyclic, then for any R' and any $\varsigma' \in \tilde{\mathcal{O}}^{|A|}$, $f^{DA}(\varsigma', R')$ is constrained efficient. It follows that DA-STB is an ex-post constrained efficient rule when $\succeq$ is T-acyclic. □

Proof of *(v)* ⇒ *(ii)*. Suppose that $\succeq$ is T-acyclic. Consider any $i, j \in N$, $\bar{a} \in A$ and R such that $\emptyset \notin U(R_i, \bar{a}) = U(R_j, \bar{a}), R_i|_{U(R_i, \bar{a})} = R_j|_{U(R_i, \bar{a})}$ and $i \sim_a j$ for all $a \in U(R_i, \bar{a})$. For each $\varsigma \in \tilde{\mathcal{O}}^{|A|}$, let ς' denote the list of orderings obtained by switching the positions of i and j, i.e., for all $a \in A$, $\varsigma'_a(i) = \varsigma_a(j)$, $\varsigma'_a(j) = \varsigma_a(i)$, and $\varsigma'_a(k) = \varsigma_a(k)$ for all $k \in N \setminus \{i, j\}$. The proof of this direction is mainly built on the following result.

Claim 2. *For any $a \in U(R_i, \bar{a})$ and $\varsigma \in \tilde{\mathcal{O}}^{|A|}$, if $f_i^{DA}(\varsigma, R) = a$ and $a P_j f_j^{DA}(\varsigma, R)$, then $f_j^{DA}(\varsigma', R) R_j a$.*

Proof. Assume to the contrary, there exist $a \in U(R_i, \bar{a})$ and $\varsigma \in \tilde{\mathcal{O}}^{|A|}$ such that $f_i^{DA}(\varsigma, R) = a$, $a P_j f_j^{DA}(\varsigma, R)$, and $a P_j f_j^{DA}(\varsigma', R)$. Let $f^{DA}(\varsigma, R) = \mu$ and $f^{DA}(\varsigma', R) = \mu'$. Since μ is stable for $(\succeq^{\varsigma}, R)$, $\varsigma_a(i) < \varsigma_a(j)$. Then $\varsigma'_a(j) < \varsigma'_a(i)$. Since μ' is stable for $(\succeq^{\varsigma'}, R)$,

$U(R_i, a) = U(R_j, a)$ and $i \sim_b j$ for all $b \in U(R_i, a)$, we have $aP_i\mu'(i)$. Consider two possible cases.

Case 1: $\mu(j)R_j\mu'(j)$.

Define a function $\nu : N \to \bar{A}$ such that for all $k \in N$, $\nu(k) = \mu(k)$ if $\mu(k)R_k\mu'(k)$, and $\nu(k) = \mu'(k)$ if $\mu'(k)P_k\mu(k)$. Then $\nu(i) = a$ and $\nu(j) = \mu(j)$. We first show that ν is a deterministic allocation, i.e., no object is over-assigned. Suppose that there exists $k \in \nu^{-1}(a) \setminus \{i\}$ such that $\mu'(k) = a$. Then $k \succ_a^{\varsigma'} j$, since $aP_j\mu'(j)$ and μ' is stable for $(\succeq^{\varsigma'}, R)$. Because i and j's priority rankings at a are switched as we move from $\succeq^{\varsigma'}$ to $\succeq^{\varsigma}$, it follows that $k \succ_a^{\varsigma} i$. Then $\mu(i) = a$ and the stability of μ for $(\succeq^{\varsigma}, R)$ imply $\mu(k)R_ka$. Then $\mu(k) = a$ since $\nu(k) = a$. Therefore, $\nu^{-1}(a) \subseteq \mu^{-1}(a)$ and $|\nu^{-1}(a)| \leq |\mu^{-1}(a)| \leq q_a$. Next, consider any $b \in A \setminus \{a\}$. b is clearly not over-assigned if $\nu^{-1}(b) \subseteq \mu^{-1}(b)$. Suppose that for some $l \in \nu^{-1}(b)$, $\mu(l) \neq b$. Then $\mu'(l) = b$ and $bP_l\mu(l)$. If there exists $m \in \nu^{-1}(b) \setminus \{l\}$ such that $\mu(m) = b$, then by the stability of μ for $(\succeq^{\varsigma}, R)$, $m \succ_b^{\varsigma} l$. We have $l \notin \{i, j\}$ since $\mu'(l)P_l\mu(l)$, and $m \neq i$ since $\mu(i) = a$. Then $\varsigma'_b(l) = \varsigma_b(l)$, and $\varsigma'_b(m) \leq \varsigma_b(m)$. It follows that $m \succ_b^{\varsigma'} l$. Since μ' is stable for $(\succeq^{\varsigma'}, R)$ and $\mu'(l) = b$, $\mu'(m)R_mb$. Given that $m \in \nu^{-1}(b)$, $\mu'(m) = b$. Therefore, $\nu^{-1}(b) \subseteq (\mu')^{-1}(b)$ and b is not over-assigned. This finishes the proof of the fact that ν is a deterministic allocation.

Consider the problem $(\succeq, R)$. By the construction, ν Pareto dominates the stable deterministic allocation μ'. Then ν is individually rational and nonwasteful.[19] If ν does not respect the priorities, then there exist some $k, l \in N$ such that $k \succ_{\nu(l)} l$ and $\nu(l)P_k\nu(k)$. For some $\mu'' \in \{\mu, \mu'\}$, $\mu''(l) = \nu(l)$. Then $k \succ_{\mu''(l)} l$ and $\mu''(l)P_k\nu(k)R_k\mu''(k)$, contradicting to the stability of μ''. In sum, ν is stable. As ν Pareto dominates μ', μ' is not constrained efficient. However, as shown in the proof of *(v)* $\Rightarrow$ *(i)*, $f^{DA}(\varsigma', \cdot)$ is a constrained efficient rule when $\succeq$ is T-acyclic. So a contradiction is reached.

Case 2: $\mu'(j)P_j\mu(j)$.

It will be shown that there exists a type-I or type-II cycle in this case. First, a finite sequence of $n+1 \geq 2$ distinct agents $(i, i_1, ..., i_n)$ is called a *chain* if the following two conditions are satisfied: (1) $\mu'(i_{k+1})P_{i_k}\mu'(i_k)$ and $\mu'(i_{k+1}) = \mu(i_k)$ for each $k \in \{0, .., n-1\}$, with $i_0 = i$; (2) $\mu'(i)P_{i_n}\mu'(i_n)$ and $\mu'(i) = \mu(i_n)$, or, $\mu'(j)P_{i_n}\mu'(i_n)$ and $\mu'(j) = \mu(i_n)$.

[19] Generally, if μ_1 is a stable deterministic allocation and μ_2 Pareto dominates μ_1, then μ_2 is individually rational and nonwasteful. While the individual rationality part is obvious, the nonwastefulness of μ_2 follows from the fact that $|\mu_1^{-1}(a)| = |\mu_2^{-1}(a)|$ for all $a \in A$, which is a part of Lemma 1 in Erdil and Ergin (2008).

We show that a chain exists. Let $N^W(b) = \left\{l \in \mu^{-1}(b) : bP_l\mu'(l)\right\}$ for each $b \in A$, and $N^W = \{l \in N : \mu(l)P_l\mu'(l)\}$. By the individual rationality of μ', $\mu(l) \in A$ for each $l \in N^W$. Hence, $N^W = \cup_{b\in A}N^W(b)$. For each $b \in A$, the nonwastefulness of μ' implies that $|(\mu')^{-1}(b) \setminus \mu^{-1}(b)| \geq |N^W(b)|$. So there exists a one-to-one function $f : N^W \to N$ such that $\mu'(f(l)) = \mu(l)$ and $\mu(f(l)) \neq \mu(l)$ for all $l \in N^W$. We know that $i \in N^W$ and $\mu'(f(i)) = a$. Since μ' is stable for $(\succeq^{\varsigma'}, R)$ and $aP_j\mu'(j)$, $f(i) \succ_a^{\varsigma'} j$. As i's priority ranking in $\succeq_a^{\varsigma}$ is the same as j's priority ranking in $\succeq_a^{\varsigma'}$, $f(i) \succ_a^{\varsigma} i$. Since $\mu(i) = a$ and μ is stable for $(\succeq^{\varsigma}, R)$, $\mu(f(i))R_{f(i)}a$. By the construction of f, $\mu(f(i)) \neq a$, so $\mu(f(i))P_{f(i)}a$. Hence, $f(i) \in N^W$. Next, consider any $l \in N^W \setminus \{i\}$. Since μ' is stable for $(\succeq^{\varsigma'}, R)$, $f(l) \succ_{\mu(l)}^{\varsigma'} l$. We know that $l \neq j$ since $j \notin N^W$. Suppose that $f(l) \neq j$ and $f(l) \neq i$, then $\varsigma_{\mu(l)}(l) = \varsigma'_{\mu(l)}(l)$ and $\varsigma_{\mu(l)}(f(l)) = \varsigma'_{\mu(l)}(f(l))$. Thus $f(l) \succ_{\mu(l)}^{\varsigma} l$. As μ is stable for $(\succeq^{\varsigma}, R)$, $\mu(f(l))R_{f(l)}\mu(l)$. Then it follows from $\mu(f(l)) \neq \mu(l)$ that $\mu(f(l))P_{f(l)}\mu(l)$. That is, $f(l) \in N^W$. In sum, it has been shown that f is a one-to-one mapping from N^W to $N^W \cup \{j\}$. This implies that a chain $(i, j_1, ..., j_m)$ can be constructed using f, where $j_{k+1} = f(j_k)$ for all $k \in \{0, ..., m-1\}$, with $j_0 = i$, and $f(j_m) \in \{i, j\}$.

Let $(i, i_1, ..., i_n)$ be the shortest chain. There exists $i^* \in \{i, j\}$ such that $\mu'(i^*)P_{i_n}\mu'(i_n)$ and $\mu'(i^*) = \mu(i_n)$. Let $\{j^*\} = \{i, j\} \setminus \{i^*\}$. We will construct a type-I or type-II cycle using the agents $i = i_0, j, i_1, ..., i_n$ and the objects $\mu'(i_1) = a, ..., \mu'(i_n), \mu'(i^*)$. We first show that these objects are distinct. Clearly $\mu'(i^*) \neq a$. If $\mu'(i_k) = a$ for some k with $1 < k \leq n$, then $(i, i_k, ..., i_n)$ is a shorter chain. Let $i_{n+1} = i^*$. If there exist k and k' such that $1 < k < k' \leq n+1$ and $\mu'(i_k) = \mu'(i_{k'})$, then $(i, i_1, ..., i_{k-1}, i_{k'}, ..., i_n)$ is a shorter chain when $k' < n+1$, and $(i, i_1, ..., i_{k-1})$ is a shorter chain when $k' = n+1$. Hence, $\mu'(i_1), ..., \mu'(i_n), \mu'(i^*)$ are distinct. Moreover, $\mu'(j^*) \neq \mu'(i_k)$ for any $k \in \{1, ..., n\}$, since $\mu'(j^*) \neq a$ and if $\mu'(j^*) = \mu'(i_k)$ for some k with $2 \leq k \leq n$, then $(i, i_1, ..., i_{k-1})$ is a shorter chain. It also follows that $i_1, ..., i_n, i^*, j^*$ are distinct. Notice that we cannot rule out the case of $\mu'(i^*) = \mu'(j^*)$.

Case 2.1: $\mu'(i^*) = \mu'(j^*)$, i.e., $\mu'(i) = \mu'(j)$. Since μ' is stable for $(\succeq, R)$ and $\mu'(i)P_{i_n}\mu'(i_n)$, $i \succeq_{\mu'(i)} i_n$ and $j \succeq_{\mu'(i)} i_n$. Given that $\mu'(j)P_j\mu(j)$ and $\mu(i_n) = \mu'(i)$, the stability of μ for $(\succeq, R)$ implies $i_n \succeq_{\mu'(i)} j$. Hence, $i \succeq_{\mu'(i)} i_n \sim_{\mu'(i)} j$. Suppose that $i \sim_{\mu'(i)} i_n \sim_{\mu'(i)} j$. By the stability of μ' for $(\succeq^{\varsigma'}, R)$, $\varsigma'_{\mu'(i)}(i) < \varsigma'_{\mu'(i)}(i_n)$. Then $\varsigma_{\mu'(i)}(j) = \varsigma'_{\mu'(i)}(i) < \varsigma'_{\mu'(i)}(i_n) = \varsigma_{\mu'(i)}(i_n)$. So $j \succ_{\mu'(i)}^{\varsigma} i_n$, contradicting to the stability of μ for $(\succeq^{\varsigma}, R)$. Therefore, we must have $i \succ_{\mu'(i)} i_n \sim_{\mu'(i)} j$. Let $N_{\mu'(i)} = \mu^{-1}(\mu'(i)) \setminus \{i_n\}$ and $N_a = \mu^{-1}(a) \setminus \{i\}$. Obviously, $N_{\mu'(i)}$ and N_a are disjoint. Since $aP_j\mu(j)$ and $\mu'(i)P_j\mu(j)$, we have $N_{\mu'(i)}, N_a \subseteq N \setminus \{i, i_n, j\}$. Then by the stability of μ for $(\succeq, R)$, $N_{\mu'(i)} \subseteq U(\succeq_{\mu'(i)}, j)$,

$N_a \subseteq U(\succeq_a, j) = U(\succeq_a, i)$, $|N_{\mu'(i)}| = q_{\mu'(i)} - 1$ and $|N_a| = q_a - 1$. Hence, we have found a type-I cycle, in which $i \succ_{\mu'(i)} i_n \sim_{\mu'(i)} j \succeq_a i$.

Case 2.2: $\mu'(i^*) \neq \mu'(j^*)$. Given that $\mu'(i_{k+1}) P_{i_k} \mu'(i_k)$ for $k = 0, ..., n$ and μ' is stable for $(\succeq, R)$, we have $i_1 \succeq_a j^* \sim_a i^*$, and $i_{k+1} \succeq_{\mu'(i_{k+1})} i_k$ for $k = n, ..., 1$. If $i_1 \sim_a j^* \sim_a i^*$, then $i_{k+1} \succ_{\mu'(i_{k+1})} i_k$ for some $k \in \{1, ..., n\}$, since otherwise the stability of μ' for $(\succeq^{\varsigma'}, R)$ implies $\varsigma'_b(i^*) < \varsigma'_b(i_n) < ... < \varsigma'_b(i_1) < \varsigma'_b(i^*)$ for any $b \in A$. So the cycle condition of a generalized type-II cycle is satisfied. If $i_1 \succ_a j^* \sim_a i^*$, then the cycle condition of a type-I cycle is satisfied. We finish the proof by establishing the same scarcity condition for either type of cycle. Let $N_k = \{l \in N : \mu'(l) = \mu'(i_k), l \neq i_k\}$ for $k = 1, ..., n+1$. Obviously, $N_1, ..., N_{n+1}$ are disjoint. Given that $\mu'(j^*) \neq \mu'(i_k)$ for any $k \in \{1, ..., n+1\}$, $N_k \subseteq N \setminus \{i^*, j^*, i_1, ..., i_n\}$ for all k. Finally, the stability of μ' for $(\succeq, R)$ implies that $N_1 \subseteq U(\succeq_a, i^*)$, $|N_1| = q_a - 1$, $N_{k+1} \subseteq U(\succeq_{\mu'(i_{k+1})}, i_k)$ and $|N_{k+1}| = q_{\mu'(i_{k+1})} - 1$ for $k = n, ..., 1$. Therefore, there exists a type-I cycle or a generalized type-II cycle. By Claim 1, there exists a type-I cycle or a type-II cycle.

In sum, it has been shown that in Case 2 there exists a type-I or type-II cycle, contradicting to the initial assumption that $\succeq$ is T-acyclic. □

Consider any $a \in U(R_i, \bar{a})$. To establish the strong symmetry of DA-STB, it is sufficient to show that for any $\varsigma \in \tilde{\mathscr{O}}^{|A|}$, $f_i^{DA}(\varsigma, R) = a$ implies $f_j^{DA}(\varsigma', R) = a$.[20]

Suppose that for some $\varsigma \in \tilde{\mathscr{O}}^{|A|}$, $f_i^{DA}(\varsigma, R) = a$. If $f_j^{DA}(\varsigma, R) = b \in U(R_i, \bar{a})$, then during the process of DA, i and j only apply to some objects in $U(R_i, \bar{a})$, and they only differ in their orderings. Thus, when their orderings are switched, their assignments are also switched. That is, $f_j^{DA}(\varsigma', R) = a$ and $f_i^{DA}(\varsigma', R) = b$.

If $f_j^{DA}(\varsigma, R) = b \notin U(R_i, \bar{a})$, then $a P_j b$. By Claim 2, $f_j^{DA}(\varsigma', R) R_j a$. Suppose that $f_j^{DA}(\varsigma', R) \neq a$, i.e., $f_j^{DA}(\varsigma', R) P_j a$. Clearly $f_j^{DA}(\varsigma', R) \in U(R_i, \bar{a})$. If $f_i^{DA}(\varsigma', R) \in U(R_i, \bar{a})$, then by a similar argument as before, i and j's assignments are switched when the orderings are changed from ς' to ς: $f_i^{DA}(\varsigma, R) = f_j^{DA}(\varsigma', R)$. It follows that $f_i^{DA}(\varsigma, R) P_i a$, contradiction. On the other hand, if $f_i^{DA}(\varsigma', R) \notin U(R_i, \bar{a})$, then $f_j^{DA}(\varsigma', R) P_i f_i^{DA}(\varsigma', R)$. Consider a change of the orderings from ς' to ς. By an argument symmetric to Claim 2, we have $f_i^{DA}(\varsigma, R) R_i f_j^{DA}(\varsigma', R)$. It follows that $f_i^{DA}(\varsigma, R) P_i a$, contradiction. □

[20] If it can be shown that $f_i^{DA}(\varsigma, R) = a$ implies $f_j^{DA}(\varsigma', R) = a$ for any $\varsigma \in \tilde{\mathscr{O}}^{|A|}$, then $|\left\{\varsigma \in \tilde{\mathscr{O}}^{|A|} : f_j^{DA}(\varsigma, R) = a\right\}| \geq |\left\{\varsigma \in \tilde{\mathscr{O}}^{|A|} : f_i^{DA}(\varsigma, R) = a\right\}|$ because the mapping $g : \tilde{\mathscr{O}}^{|A|} \to \tilde{\mathscr{O}}^{|A|}$, where $g(\varsigma) = \varsigma'$ for all $\varsigma \in \tilde{\mathscr{O}}^{|A|}$, is one-to-one. By symmetric arguments, $|\left\{\varsigma \in \tilde{\mathscr{O}}^{|A|} : f_i^{DA}(\varsigma, R) = a\right\}| \geq |\left\{\varsigma \in \tilde{\mathscr{O}}^{|A|} : f_j^{DA}(\varsigma, R) = a\right\}|$. Hence $f_{ia}^{DA\text{-}STB}(R) = f_{ja}^{DA\text{-}STB}(R)$.

Proof of *(iii)* $\Rightarrow$ *(v)*. Suppose that there exists a strategy-proof, ex-post constrained efficient, symmetric and locally envy-free rule f, but $\succeq$ is not T-acyclic.

Case 1: there exists a type-I cycle.

Let this cycle consist of distinct $i_1, ..., i_n \in N$, $n \geq 3$, and distinct $a_1, ..., a_{n-1} \in A$ such that: (1) $i_1 \succ_{a_1} i_2 \sim_{a_1} i_3$, and $i_{k+1} \succeq_{a_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$; (2) there exist $n-1$ mutually disjoint sets $N_1, ..., N_{n-1} \subseteq N \setminus \{i_1, ..., i_n\}$ such that $N_k \subseteq U(\succeq_{a_k}, i_{k+2})$ and $|N_k| = q_{a_k} - 1$ for each k with $1 \leq k \leq n-1$. Consider the following two preference profiles.[21]

$$R: \quad \begin{array}{ll} i_1: & a_{n-1}, a_1, \emptyset \\ i_2: & a_1, \emptyset \\ i_k: & a_{k-2}, a_{k-1}, \emptyset \quad \text{for } k = 3, .., n \\ i \in N_k: & a_k, \emptyset \quad \text{for } k = 1, .., n-1 \end{array}$$

$$R': \quad \begin{array}{ll} i_1: & a_{n-1}, \emptyset \\ i_2: & a_1, \emptyset \\ i_k: & a_{k-2}, \emptyset \quad \text{for } k = 3, .., n \\ i \in N_k: & a_k, \emptyset \quad \text{for } k = 1, .., n-1 \end{array}$$

Since f is ex-post constrained efficient, there exists a lottery λ that induces $f(R)$ and each $\mu \in \mathscr{S}(\lambda)$ is constrained efficient.

First, we show that $f_{i_1 a_1}(R) = 0$. Assume to the contrary, $f_{i_1 a_1}(R) > 0$. Then for some $\mu \in \mathscr{S}(\lambda)$, $\mu(i_1) = a_1$. If $\mu(i_2) = a_1$, then there exists $i \in N_1$ such that $\mu(i) = \emptyset$. By stability, $i \sim_{a_1} i_2 \sim_{a_1} i_3$. The other possible case is that $\mu(i_2) \neq a_1$. Therefore, in sum, there must exist some $j \in \{i_2\} \cup N_1$ such that $\mu(j) = \emptyset$ and $j \sim_{a_1} i_3$. Consider the deterministic allocation ν in which $\nu(j) = \emptyset$ and every agent except j is assigned her top choice. ν is well defined given the construction of R. It follows from $j \sim_{a_1} i_3$ that $j' \succeq_{a_1} j$ for all $j' \in \nu^{-1}(a_1)$. So ν is stable. A contradiction is reached since ν Pareto dominates the constrained efficient deterministic allocation μ.

Second, we show that $f_{i_1 a_{n-1}}(R) < 1$. Assume to the contrary, $f_{i_1 a_{n-1}}(R) = 1$. Then by strategy-proofness, $f_{i_1 a_{n-1}}(R'_{i_1}, R_{-i_1}) = 1$. Consider any $i \in N_{n-1}$. If $i \succ_{a_{n-1}} i_1$, then by ex-post stability, $f_{i a_{n-1}}(R'_{i_1}, R_{-i_1}) = 1$. If $i \sim_{a_{n-1}} i_1$, then by symmetry, $f_{i a_{n-1}}(R'_{i_1}, R_{-i_1}) = 1$. It follows that $f_{i_n a_{n-1}}(R'_{i_1}, R_{-i_1}) = 0$. Since $i_n \succeq_{a_{n-1}} i_1$, $i_n \succ_{a_{n-1}} i_1$ or $i_n \sim_{a_{n-1}} i_1$. In

[21] Any unlisted agent is assumed to rank her outside option at the top. Then such an agent will be assigned her outside option with probability one under an individually rational rule.

the former case, by ex-post stability, $f_{i_n a_{n-2}}(R'_{i_1}, R_{-i_1}) = 1$. In the latter case, by local envy-freeness, we also have $f_{i_n a_{n-2}}(R'_{i_1}, R_{-i_1}) = 1$. By similar arguments, it can be shown that $f_{i_n a_{n-2}}(R'_{i_1}, R_{-i_1}) = 1$ implies $f_{i_n a_{n-2}}(R'_{i_1}, R'_{i_n}, R_{-\{i_1,i_n\}}) = 1$, which further implies $f_{i_{n-1} a_{n-3}}(R'_{i_1}, R'_{i_n}, R_{-\{i_1,i_n\}}) = 1$ (when $n > 3$). Continuing in this fashion, eventually we have $f_{i_3 a_1}(R') = 1$. Then by symmetry and ex-post stability, $f_{i_2 a_1}(R') = 1$ and $f_{i a_1}(R') = 1$ for all $i \in N_1$. A contradiction is reached since at least $q_{a_1} + 1$ agents are assigned a_1 with probability one.

Finally, given that $f_{i_1 a_1}(R) = 0$ and $f_{i_1 a_{n-1}}(R) < 1$, $f_{i_1 \emptyset}(R) > 0$. So for some $\mu' \in \mathscr{S}(\lambda)$, $\mu'(i_1) = \emptyset$. Since μ' is stable and $i_1 \succ_{a_1} i_2 \sim_{a_1} i_3$, $\mu'(i_2) \neq a_1$ and $\mu'(i_3) \neq a_1$. It follows that $(\mu')^{-1}(a_1) \subseteq N_1$, contradicting to the nonwastefulness of μ'.

Case 2: there exists a type-II cycle.

We first refine the notion of a type-II cycle. A **type-II* cycle** consists of distinct $i, j, k \in N$ and $a, b \in A$ such that the following conditions are satisfied: (Cycle) $i \sim_a j \sim_a k \succ_b i$ and $j \succ_b i$; (Scarcity) there exist (possibly empty) disjoint sets $N_a, N_b \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq U(\succeq_a, i)$, $N_a \subseteq SU(\succeq_b, i)$, $N_b \subseteq SU(\succeq_b, i)$, $|N_a| = q_a - 1$ and $|N_b| = q_b - 1$.

Claim 3. *If there is a type-II cycle but not any type-I cycle, then there is a type-II* cycle.*

Proof. Suppose that there does not exist any type-I cycle, and there exists a type-II cycle that consists of distinct $i, j, k \in N$ and $a, b \in A$ such that: (1) $i \sim_a j \sim_a k \succ_b i$; (2) there exist disjoint sets $N_a, N_b \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq U(\succeq_a, i)$, $N_b \subseteq U(\succeq_b, i)$, $|N_a| = q_a - 1$ and $|N_b| = q_b - 1$. If $j \sim_b i$, then there is a type-I cycle since $k \succ_b i \sim_b j \succeq_a k$, $N_b \subseteq U(\succeq_b, i)$ and $N_a \subseteq U(\succeq_a, k)$. So we have either $j \succ_b i$ or $i \succ_b j$. Suppose $j \succ_b i$. We will show that a type-II* cycle exists in this case. The other case can be shown similarly by switching the roles of i and j.

First, $N_b \subseteq SU(\succeq_b, i)$. If this is not true, there exists $l \in N_b$ such that $l \sim_b i$. A type-I cycle is found since $k \succ_b l \sim_b i \succeq_a k$ and the scarcity condition is satisfied by $(N_b \setminus \{l\}) \cup \{j\}$ and N_a.

Second, if $l \in N_a$ and $l \succ_a i$, $l \succ_b i$. To see this, suppose $l \in N_a$, $l \succ_a i$ and $i \succeq_b l$. Then there is a type-I cycle: $l \succ_a j \sim_a i \succeq_b l$ and the scarcity condition is satisfied by $(N_a \setminus \{l\}) \cup \{k\}$ and N_b.

Third, if $l \in N_a$ and $l \sim_a i$, $l \not\sim_b i$. If this is not true, there is a type-I cycle: $k \succ_b l \sim_b i \succeq_a k$ and the scarcity condition is satisfied by N_b and $(N_a \setminus \{l\}) \cup \{j\}$.

Fourth, if $l, l' \in N_a$, $l \neq l'$, $l \sim_a l' \sim_a i$, $i \succ_b l$ and $i \succ_b l'$, then $l \not\sim_b l'$. If this is not true, there is a type-I cycle: $i \succ_b l \sim_b l' \succeq_a i$ and the scarcity condition is satisfied by N_b and $(N_a \setminus \{l, l'\}) \cup \{j, k\}$.

Therefore, if $N^* = \{l \in N_a : l \sim_a i, i \succ_b l\}$ is empty, then by the first three results the type-II cycle that consists of $i, j, k \in N$ and $a, b \in A$ is also a type-II* cycle. If N^* is nonempty, by the fourth result, we can find $l \in N^*$ such that $l' \succ_b l$ for any $l' \in N^* \setminus \{l\}$. Then i, j, l and a, b form a type-II* cycle: $l \sim_a i \sim_a j \succ_b l$, $i \succ_b l$, and the scarcity condition is satisfied by $(N_a \setminus \{l\}) \cup \{k\}$ and N_b. □

In light of Claim 3, it is sufficient to consider a type-II* cycle. Let this cycle consist of $i, j, k \in N$ and $a, b \in A$ such that: (1) $i \sim_a j \sim_a k \succ_b i$ and $j \succ_b i$; (2) there exist disjoint $N_a, N_b \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq U(\succeq_a, i)$, $N_a \subseteq SU(\succeq_b, i)$, $N_b \subseteq SU(\succeq_b, i)$, $|N_a| = q_a - 1$ and $|N_b| = q_b - 1$. Let $N_a^* = \{l \in N_a : l \sim_a i\}$ and $q = q_a - |N_a \setminus N_a^*|$. Then $|N_a^*| = q - 1$. In the following analysis, we will restrict attention to preference profiles in which a is only acceptable to some agents in $\{i, j, k\} \cup N_a$, and the agents in $N_a \setminus N_a^*$ rank a at the top. By the ex-post stability of f, each agent in $N_a \setminus N_a^*$ is always assigned a with probability one. Hence, $q \geq 1$ is essentially the number of copies of a available for $\{i, j, k\} \cup N_a^*$. Consider the following preference profiles.

	$i:$	$a, \emptyset$		$i:$	$a, b, \emptyset$		$i:$	$b, a, \emptyset$
	$j:$	$a, \emptyset$		$j:$	$a, b, \emptyset$		$j:$	$a, \emptyset$
$R^1:$	$k:$	$a, \emptyset$	$R^2:$	$k:$	$a, b, \emptyset$	$R^3:$	$k:$	$a, b, \emptyset$
	$N_a^*:$	$a, \emptyset$		$N_a^*:$	$a, b, \emptyset$		$N_a^*:$	$a, b, \emptyset$
	$N_b:$	$b, \emptyset$		$N_b:$	$b, \emptyset$		$N_b:$	$b, \emptyset$
	$N_a \setminus N_a^*:$	$a, \emptyset$		$N_a \setminus N_a^*:$	$a, \emptyset$		$N_a \setminus N_a^*:$	$a, \emptyset$

At R^1, by individual rationality and ex-post nonwastefulness, $\sum_{l \in \{i,j,k\} \cup N_a^*} f_{la}(R^1) = q$.[22] Then by symmetry, $f_{la}(R^1) = \frac{q}{q+2}$ for all $l \in \{i, j, k\} \cup N_a^*$. We use induction to show that the allocation of a is the same at R^2. Consider the following statement: for any $N' \subseteq \{i, j, k\} \cup N_a^*$ with $|N'| = n$, $f_{la}(R^2_{N'}, R^1_{-N'}) = \frac{q}{q+2}$ for all $l \in \{i, j, k\} \cup N_a^*$. It is obviously true if $n = 0$. Suppose that the statement is true for n with $0 \leq n \leq q+1$. Consider any $N' \subseteq \{i, j, k\} \cup N_a^*$ with $|N'| = n+1$ and any $l \in N'$. Then $f_{la}(R^2_{N' \setminus \{l\}}, R^1_{-N' \setminus \{l\}}) = \frac{q}{q+2}$. By

[22] Formally, an allocation M is ex-post nonwasteful if some lottery λ induces M and each $\mu \in \mathscr{S}(\lambda)$ is nonwasteful. It is worth mentioning that an ex-post nonwasteful allocation M may involve "waste" ex-ante, i.e., there may exist $i \in N$ and $a \in A$ such that $\sum_{b \in U(R_i, a)} M_{ib} < 1$ and $\sum_{j \in N} M_{ja} < q_a$. However, it can be easily shown that generally if an allocation M is ex-post nonwasteful and there are at least q_a agents rank some object $a \in A$ at the top, then $\sum_{i \in N} M_{ia} = q_a$.

strategy-proofness, $f_{la}(R^2_{N'}, R^1_{-N'}) = \frac{q}{q+2}$. Notice that the choice of l is arbitrary within the set N'. So $f_{l'a}(R^2_{N'}, R^1_{-N'}) = \frac{q}{q+2}$ for all $l' \in N'$. By individual rationality and ex-post nonwastefulness, $\sum_{l' \in (\{i,j,k\} \cup N^*_a) \setminus N'} f_{l'a}(R^2_{N'}, R^1_{-N'}) = q - \frac{q}{q+2}|N'|$. Then the symmetry of f with respect to $(\{i,j,k\} \cup N^*_a) \setminus N'$ implies $f_{l'a}(R^2_{N'}, R^1_{-N'}) = \frac{q}{q+2}$ for all $l' \in \{i,j,k\} \cup N^*_a$. That is, the statement is true for $n+1$. Therefore, $f_{l'a}(R^2) = \frac{q}{q+2}$ for all $l' \in \{i,j,k\} \cup N^*_a$.

Let μ be any stable deterministic allocation for $(\succeq, R^2)$. Since $l \succ_b i$ for all $l \in \{j,k\} \cup N^*_a \cup N_b$, if $\mu(i) = b$, then $\mu(l) = b$ for all $l \in N_b$, and $\mu(l) = a$ for all $l \in \{j,k\} \cup N^*_a$, which is not possible. So by ex-post stability, $f_{ib}(R^2) = 0$. By the same reasoning, $f_{ib}(R^3_i, R^2_{-i}) = 0$. Then the strategy-proofness of f implies $f_{ia}(R^3_i, R^2_{-i}) = f_{ia}(R^2) = \frac{q}{q+2}$. It follows that for some $l \in \{j,k\} \cup N^*_a$, $f_{la}(R^3_i, R^2_{-i}) \leq \frac{q}{q+2}$. Without loss of generality, suppose that $f_{ja}(R^3_i, R^2_{-i}) \leq \frac{q}{q+2}$. Then by strategy-proofness,

$$f_{ja}(R^3) = f_{ja}(R^3_i, R^2_{-i}) \leq \frac{q}{q+2} \tag{1}$$

Consider R_3. We argue that $f_{ib}(R^3) + f_{ja}(R^3) \geq 1$. Assume to the contrary, $f_{ib}(R^3) + f_{ja}(R^3) < 1$. Let λ be a lottery that induces $f(R^3)$ such that each $\mu \in \mathscr{S}(\lambda)$ is constrained efficient. Then there exists $\mu \in \mathscr{S}(\lambda)$ such that $\mu(i) \neq b$ and $\mu(j) \neq a$. By individual rationality, $\mu(j) = \emptyset$. Consider the deterministic allocation ν in which $\nu(j) = \emptyset$ and every other agent is assigned her top choice. It is easy to see that ν is stable and it Pareto dominates μ, contradicting to the constrained efficiency of μ. Therefore, $f_{ib}(R^3) + f_{ja}(R^3) \geq 1$. Then it follows from (1) that

$$f_{ib}(R^3) \geq \frac{2}{q+2} \tag{2}$$

Finally, consider the following preference profiles.

R^4 :	i :	$b, \emptyset$	R^5 :	i :	$b, \emptyset$
	j :	$a, \emptyset$		j :	$a, \emptyset$
	k :	$a, \emptyset$		k :	$a, b, \emptyset$
	N^*_a :	$a, \emptyset$		N^*_a :	$a, b, \emptyset$
	N_b :	$b, \emptyset$		N_b :	$b, \emptyset$
	$N_a \setminus N^*_a$:	$a, \emptyset$		$N_a \setminus N^*_a$:	$a, \emptyset$

At R^4, by individual rationality and ex-post nonwastefulness, $\sum_{l \in \{j,k\} \cup N^*_a} f_{la}(R^4) = q$. Then by symmetry, $f_{la}(R^4) = \frac{q}{q+1}$ for all $l \in \{j,k\} \cup N^*_a$. Similar to the previous discussion about moving from R^1 to R^2, we can use induction to show that $f_{la}(R^5) = \frac{q}{q+1}$ for all

$l \in \{j,k\} \cup N_a^*$. Next, consider any stable deterministic allocation μ for $(\succeq, R^5)$ and any $l \in \{k\} \cup N_a^* \cup N_b$. Suppose that $\mu(l) = \emptyset$. Then by stability $\mu(i) = \emptyset$. Since the q copies of a and q_b copies of b can only be allocated to $\{i,j,k\} \cup N_a^* \cup N_b$ at μ and $|\{i,j,k\} \cup N_a^* \cup N_b| = q + q_b + 1$, there is at least one copy of a or one copy of b that is unassigned. This implies that μ is wasteful because $q+1$ agents in $\{i,j,k\} \cup N_a^* \cup N_b$ rank a at the top and q_b agents in this set rank b at the top. Therefore, $\mu(l') \neq \emptyset$ for any $l' \in \{k\} \cup N_a^* \cup N_b$. Then by ex-post stability, $f_{l'b}(R^5) = 1$ for all $l' \in N_b$, and $f_{l'b}(R^5) = 1 - \frac{q}{q+1} = \frac{1}{q+1}$ for all $l' \in \{k\} \cup N_a^*$. Moreover, $\sum_{l' \in \{i,k\} \cup N_a^* \cup N_b} f_{l'b}(R^5) = q_b$. Hence,

$$\begin{aligned} f_{ib}(R^5) &= q_b - \sum_{l' \in \{k\} \cup N_a^*} f_{l'b}(R^5) - \sum_{l' \in N_b} f_{l'b}(R^5) \\ &= q_b - \frac{q}{q+1} - (q_b - 1) \\ &= \frac{1}{q+1} \\ &< \frac{2}{q+2} \end{aligned} \tag{3}$$

Comparing (2) and (3), f is not strategy-proof, contradiction. □

Proof of *(iv)* ⇒ *(v)*. Suppose that there exists a strategy-proof, ex-post stable, symmetric at the top and locally envy-free rule f, but $\succeq$ is not T-acyclic. In light of Claim 3, we consider the following two cases.

Case 1: There is a type-I cycle.

Let this cycle consist of distinct $i_1, ..., i_n \in N$, $n \geq 3$, and distinct $a_1, ..., a_{n-1} \in A$ such that: (1) $i_1 \succ_{a_1} i_2 \sim_{a_1} i_3$, and $i_{k+1} \succeq_{a_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$; (2) there exist $n-1$ mutually disjoint sets $N_1, ..., N_{n-1} \subseteq N \setminus \{i_1, ..., i_n\}$ such that $N_k \subseteq U(\succeq_{a_k}, i_{k+2})$ and $|N_k| = q_{a_k} - 1$ for each k with $1 \leq k \leq n-1$. Consider the following preference profile.

$$\begin{array}{lll} & i_1: & a_{n-1}, a_1, \emptyset \\ & i_2: & a_1, \emptyset \\ R: & i_3: & a_1, \emptyset \\ & i_k: & a_{k-2}, a_{k-1}, \emptyset \quad \text{for } k = 4, .., n \ \ (\text{if } n \geq 4) \\ & N_k: & a_k, \emptyset \quad \text{for } k = 1, .., n-1 \end{array}$$

Denote $N_1^* = \{i \in N_1 : i \sim_{a_1} i_2\}$. Let μ be any stable deterministic allocation for $(\succeq, R)$. Then $\mu(i) = a_1$ for all $i \in N_1 \setminus N_1^*$. If $n = 3$, clearly $\mu(i_1) = a_2$. Suppose $n \geq 4$. By nonwastefulness and individual rationality, $\mu(i_4) = a_2$, then $\mu(i_5) = a_3, ..., \mu(i_{n+1}) = a_{n-1}$. Therefore, by ex-post stability, $f_{i_1 a_{n-1}}(R) = 1$ and $f_{i_1 a_1}(R) = 0$. Moreover, $f_{ia_1}(R) = 1$ for all $i \in N_1 \setminus N_1^*$, and $\sum_{i \in \{i_2, i_3\} \cup N_1^*} f_{ia_1}(R) = q_{a_1} - |N_1 \setminus N_1^*| \geq 1$. It follows from symmetry at the top that $f_{ia_1}(R) = f_{i_3 a_1}(R)$ for all $i \in \{i_2, i_3\} \cup N_1^*$.

Next, define $R'_{i_3} : a_1, a_2, \emptyset$. Let $R' = (R'_{i_3}, R_{-i_3})$. By strategy-proofness, $f_{i_3 a_1}(R') = f_{i_3 a_1}(R)$. By symmetry at the top, $f_{ia_1}(R') = f_{i_3 a_1}(R)$ for all $i \in \{i_2, i_3\} \cup N_1^*$. The ex-post stability of f implies $f_{ia_1}(R') = 1$ for all $i \in N_1 \setminus N_1^*$. Hence, the allocation of a_1 remains the same at R' and $f_{i_1 a_1}(R') = 0$. Let μ be any stable deterministic allocation for $(\succeq, R')$. If $\mu(i_1) = \emptyset$, then $\mu(i_2) \neq a_1$ and $\mu(i_3) \neq a_1$. This implies that a_1 is wasted. So by ex-post stability, $f_{i_1 \emptyset}(R') = 0$. Therefore, $f_{i_1 a_{n-1}}(R') = 1$. Consider any $i \in N_{n-1}$. If $i \succ_{a_{n-1}} i_1$, then by ex-post stability $f_{ia_{n-1}}(R') = 1$. If $i \sim_{a_{n-1}} i_1$, then by symmetry at the top $f_{ia_{n-1}}(R') = 1$. It follows that $f_{i_n a_{n-1}}(R') = 0$. Since $i_n \succeq_{a_{n-1}} i_1$, by ex-post stability (when $i_n \succ_{a_{n-1}} i_1$) or local envy-freeness (when $i_n \sim_{a_{n-1}} i_1$), we have $f_{i_n a_{n-2}}(R') = 1$. By similar arguments, it can be shown that $f_{i_{n-1} a_{n-3}}(R') = 1$ (if $n \geq 4$) and so on. Eventually this leads to $f_{i_3 a_1}(R') = 1$. Then it follows from previous discussion that $f_{ia_1}(R') = 1$ for all $i \in \{i_2, i_3\} \cup N_1$, which is impossible.

Case 2: There is a type-II* cycle.

Let this cycle consist of $i, j, k \in N$ and $a, b \in A$ such that: (1) $i \sim_a j \sim_a k \succ_b i$ and $j \succ_b i$; (2) there exist disjoint $N_a, N_b \subseteq N \setminus \{i, j, k\}$ such that $N_a \subseteq U(\succeq_a, i)$, $N_a \subseteq SU(\succeq_b, i)$, $N_b \subseteq SU(\succeq_b, i)$, $|N_a| = q_a - 1$ and $|N_b| = q_b - 1$. Similar to Case 2 in the proof of $(iii) \Rightarrow (v)$, let $N_a^* = \{l \in N_a : l \sim_a i\}$. Then in the following analysis, each $l \in N_a \setminus N_a^*$ will always be assigned a with probability one. Let $q = q_a - |N_a \setminus N_a^*|$, then $|N_a^*| = q - 1 \geq 0$. Consider the following preference profiles.

$$R: \begin{array}{ll} i: & b, a, \emptyset \\ j: & a, \emptyset \\ k: & a, \emptyset \\ N_a^*: & a, \emptyset \\ N_b: & b, \emptyset \\ N_a \setminus N_a^*: & a, \emptyset \end{array} \qquad R': \begin{array}{ll} i: & b, a, \emptyset \\ j: & a, b, \emptyset \\ k: & a, b, \emptyset \\ N_a^*: & a, b, \emptyset \\ N_b: & b, \emptyset \\ N_a \setminus N_a^*: & a, \emptyset \end{array}$$

By individual rationality and ex-post nonwastefulness, we have $f_{ib}(R) = 1$, $f_{ia}(R) = 0$ and $\sum_{l \in \{j,k\} \cup N_a^*} f_{la}(R) = q$. By symmetry at the top, $f_{la}(R) = \frac{q}{q+1}$ for all $l \in \{j, k\} \cup N_a^*$.

By strategy-proofness, $f_{ja}(R'_j, R_{-j}) = \frac{q}{q+1}$. By symmetry at the top, $f_{la}(R'_j, R_{-j}) = \frac{q}{q+1}$ for all $l \in \{j,k\} \cup N_a^*$. We can use the same reasoning to show $f_{la}(R'_{\{j,k\}}, R_{-\{j,k\}}) = \frac{q}{q+1}$ for all $l \in \{j,k\} \cup N_a^*$, and so on. In the end, we have $f_{la}(R') = \frac{q}{q+1}$ for all $l \in \{j,k\} \cup N_a^*$. It follows that $f_{ia}(R') = 0$. Next, let μ be any stable deterministic allocation for $(\succeq, R')$. If $\mu(i) = b$, $\mu(l) = b$ for all $l \in N_b$. Then respecting priorities further requires $\mu(l) = a$ for all $l \in \{j,k\} \cup N_a^*$, which is not possible. So by ex-post stability, $f_{ib}(R') = 0$. Given that $f_{ia}(R') = f_{ib}(R') = 0$, i can manipulate by reporting $R''_i : a, \emptyset$, since by symmetry at the top $f_{ia}(R''_i, R'_{-i}) > 0$. This contradicts to the strategy-proofness of f. □

Proof of Theorem 2

It is already known that $(iii) \Rightarrow (i)$ and $(i) \Rightarrow (ii)$, we show $(ii) \Rightarrow (iii)$. Assume to the contrary, there exists a strategy-proof, ex-post stable and efficient, symmetric and locally envy-free rule f, but $\succeq$ is not strongly T-acyclic. Then by Theorem 1, $\succeq$ is not strongly acyclic. That is, there exists a weak cycle that consists of distinct $i, j, k \in N$ and $a, b \in A$ such that: (1) $i \succeq_a j \succ_a k \succeq_b i$; (2) there exist disjoint $N_a, N_b \subseteq N \setminus \{i,j,k\}$ such that $N_a \subseteq U(\succeq_a, j)$, $N_b \subseteq U(\succeq_b, i)$, $|N_a| = q_a - 1$ and $|N_b| = q_b - 1$. Consider the following preference profiles.

$$
R: \begin{array}{ll} i: & b, a, \emptyset \\ j: & a, \emptyset \\ k: & a, b, \emptyset \\ N_a: & a, \emptyset \\ N_b: & b, \emptyset \end{array}
\qquad
R': \begin{array}{ll} i: & b, \emptyset \\ j: & a, \emptyset \\ k: & a, b, \emptyset \\ N_a: & a, \emptyset \\ N_b: & b, \emptyset \end{array}
$$

Let μ be stable and efficient for $(\succeq, R)$. Suppose $\mu(i) = a$. Then by individual rationality and nonwastefulness $\mu(k) = b$, which contradicts efficiency. So by ex-post stability and efficiency, $f_{ia}(R) = 0$.

Next, we show $f_{ib}(R) < 1$. Suppose that $f_{ib}(R) = 1$. Then by strategy-proofness $f_{ib}(R') = 1$. Consider any $l \in N_b$. If $l \succ_b i$, then by ex-post stability $f_{lb}(R') = 1$. If $l \sim_b i$, then by symmetry $f_{lb}(R') = 1$. Therefore, $f_{kb}(R') = 0$. As $k \succeq_b i$, by ex-post stability (when $k \succ_b i$) or local envy-freeness (when $k \sim_b i$), $f_{ka}(R') = 1$. Since $(N_a \cup \{j\}) \subseteq SU(\succeq_a, k)$, the ex-post stability of f implies $f_{la}(R') = 1$ for all $l \in N_a \cup \{j\}$, which is clearly impossible.

Given that $f_{ia}(R) = 0$ and $f_{ib}(R) < 1$, we have $f_{i\emptyset}(R) > 0$. Then by ex-post stability (when $i \succ_a j$) or local envy-freeness (when $i \sim_a j$), $f_{ja}(R) < 1$. Let λ be a lottery

that induces $f(R)$ such that each $\mu \in \mathscr{S}(\lambda)$ is stable and efficient. Then there exists $\mu \in \mathscr{S}(\lambda)$ with $\mu(j) = \emptyset$ and $\mu(i) \neq a$. By stability, $\mu(k) \neq a$. Therefore, by individual rationality a is assigned to at most $|N_a|$ agents at μ, contradicting to the nonwastefulness of μ. □

Appendix B: Independence of axioms

Theorem 1 and 2 include the following three impossibility results:

I.1. *If $\succeq$ is not T-acyclic, there does not exist a strategy-proof, ex-post constrained efficient, symmetric and locally envy-free rule.*
I.2. *If $\succeq$ is not T-acyclic, there does not exist a strategy-proof, ex-post stable, symmetric at the top and locally envy-free rule.*
I.3. *If $\succeq$ is not strongly T-acyclic, there does not exist a strategy-proof, ex-post stable and efficient, symmetric and locally envy-free rule.*

In this section we briefly discuss the independence of axioms in these results. We keep the following assumption, $A.1$, throughout this section. If $A.1$ is not satisfied, $\succeq$ is guaranteed to be strongly T-acyclic.

A.1. *There exist distinct $a, b \in A$ such that $|N| \geq q_a + q_b + 1$.*

Stronger impossibility results can be obtained in the special case of one-to-one allocation problems. In fact, those stronger results can be established as long as the objects in a cycle have unit capacity.

A.2. *There exist distinct $a, b \in A$ such that $|N| \geq q_a + q_b + 1$ and $q_a \geq 2$.*

It can be easily seen that, when $A.2$ is not satisfied, a type-I, type-II or weak cycle can only involve objects with unit capacity. In this case, symmetry can be dropped from $I.1$, and both symmetry and strategy-proofness can be dropped from $I.3$.

I.1*. *Suppose that $A.2$ is not satisfied. If $\succeq$ is not T-acyclic, there does not exist a strategy-proof, ex-post constrained efficient and locally envy-free rule.*

Proof. Suppose that $A.2$ is not satisfied and $\succeq$ is not T-acyclic.

First, consider the case in which there exists a type-I cycle. We use arguments similar to those in Case 1, proof of $(iii) \Rightarrow (v)$, Theorem 1, to show that there does not exist an ex-post constrained efficient and locally envy-free rule. Assume to the contrary, there exists such a rule f. Let the type-I cycle consist of distinct $i_1, ..., i_n \in N$, $n \geq 3$, and distinct $a_1, ..., a_{n-1} \in A$ such that $q_{a_k} = 1$ for all k with $1 \leq k \leq n-1$, $i_1 \succ_{a_1} i_2 \sim_{a_1} i_3$, and $i_{k+1} \succeq_{a_k} i_{k+2}$ for all k with $2 \leq k \leq n-1$, where $i_{n+1} = i_1$. Consider the following preference profile.

$$R: \quad \begin{array}{ll} i_1: & a_{n-1}, a_1, \emptyset \\ i_2: & a_1, \emptyset \\ i_k: & a_{k-2}, a_{k-1}, \emptyset \quad \text{for } k = 3, .., n \end{array}$$

Let μ be a deterministic allocation for $(\succeq, R)$. If $\mu(i_1) = a_1$, then μ is Pareto dominated by the stable allocation ν in which $\nu(i_2) = \emptyset$ and every other agent is assigned her top choice. So by ex-post constrained efficiency, $f_{i_1 a_1}(R) = 0$. Next, suppose $f_{i_1 a_{n-1}}(R) = 1$. Then $f_{i_n a_{n-1}}(R) = 0$. By ex-post stability (when $i_n \succ_{a_{n-1}} i_1$) or local envy-freeness (when $i_n \sim_{a_{n-1}} i_1$), $f_{i_n a_{n-2}}(R) = 1$. Continuing in this fashion, in the end we have $f_{i_3 a_1}(R) = 1$. Then $f_{i_2 \emptyset}(R) = 1$, contradicting to local envy-freeness. Finally, given that $f_{i_1 a_1}(R) = 0$ and $f_{i_1 a_{n-1}}(R) < 1$, $f_{i_1 \emptyset}(R) > 0$. Let λ be a lottery that induces $f(R)$ such that each $\mu' \in \mathscr{S}(\lambda)$ is constrained efficient. There there exists $\mu' \in \mathscr{S}(\lambda)$ such that $\mu'(i_1) = \emptyset$. Then by stability, $\mu'(i_2) \neq a_1$ and $\mu'(i_3) \neq a_1$. Therefore, a contradiction is reached since a_1 is wasted.

Second, consider the case in which there exists a type-II* cycle. So there exist distinct $i, j, k \in N$ and $a, b \in A$ such that $q_a = q_b = 1$, $i \sim_a j \sim_a k \succ_b i$ and $j \succ_b i$. Without loss of generality, let $k \succeq_b j$. To obtain a contradiction, assume that there exists a strategy-proof, ex-post constrained efficient and locally envy-free rule f. Consider the following preference profiles.

$$R^1: \begin{array}{ll} i: & b, a, \emptyset \\ j: & b, a, \emptyset \\ k: & a, b, \emptyset \end{array} \qquad R^2: \begin{array}{ll} i: & b, a, \emptyset \\ j: & a, b, \emptyset \\ k: & a, b, \emptyset \end{array} \qquad R^3: \begin{array}{ll} i: & b, a, \emptyset \\ j: & a, \emptyset \\ k: & a, \emptyset \end{array}$$

Consider any constrained efficient allocation μ for $(\succeq, R^1)$. Suppose that $\mu(j) = a$. Then by stability $\mu(k) = b$. But μ is Pareto dominated by the stable allocation ν in which $\nu(j) = b$ and $\nu(k) = a$, contradiction. So by ex-post constrained efficiency, $f_{ja}(R^1) = 0$. Next, suppose that $f_{jb}(R^1) = 1$. Then $f_{kb}(R^1) = 0$. By ex-post stability (when $k \succ_b j$) or

local envy-freeness (when $k \sim_b j$), $f_{ka}(R^1) = 1$. Then $f_{i\emptyset}(R^1) = 1$, contradicting to local envy-freeness. Therefore, $f_{jb}(R^1) + f_{ja}(R^1) < 1$.

By strategy-proofness, $f_{jb}(R^2) + f_{ja}(R^2) < 1$. Then by ex-post nonwastefulness, $f_{ib}(R^2) + f_{ia}(R^2) > 0$. It can be easily seen that for any stable deterministic allocation μ for $(\succeq, R^2)$, $\mu(i) \neq b$. So by ex-post stability $f_{ib}(R^2) = 0$. Therefore, $f_{ia}(R^2) > 0$.

Let μ be any constrained efficient allocation for $(\succeq, (R^3_j, R^2_{-j}))$. Suppose that $\mu(i) = a$. Then by stability $\mu(k) = b$. But μ is Pareto dominated by the stable allocation ν in which $\nu(i) = b$ and $\nu(k) = a$, contradiction. So by ex-post constrained efficiency, $f_{ia}(R^3_j, R^2_{-j}) = 0$. Similarly, it can be shown that $f_{ia}(R^3_k, R^2_{-k}) = 0$.

By strategy-proofness, $f_{ja}(R^3_j, R^2_{-j}) = f_{ja}(R^2)$. Then $f_{ia}(R^2) > 0$ and $f_{ia}(R^3_j, R^2_{-j}) = 0$, together with the ex-post nonwastefulness of f, imply $f_{ka}(R^3_j, R^2_{-j}) = f_{ka}(R^2) + f_{ia}(R^2)$. Similarly, it can be shown that $f_{ja}(R^3_k, R^2_{-k}) = f_{ja}(R^2) + f_{ia}(R^2)$.

Finally, by strategy-proofness, $f_{ja}(R^3) = f_{ja}(R^3_k, R^2_{-k}) = f_{ja}(R^2) + f_{ia}(R^2)$, and $f_{ka}(R^3) = f_{ka}(R^3_j, R^2_{-j}) = f_{ka}(R^2) + f_{ia}(R^2)$. A contradiction is reached since $f_{ja}(R^3) + f_{ka}(R^3) = 2f_{ia}(R^2) + f_{ja}(R^2) + f_{ka}(R^2) > 1$. □

I.3*. *Suppose that A.2 is not satisfied. If $\succeq$ is not strongly T-acyclic, there does not exist an ex-post stable and efficient, and locally envy-free rule.*

Proof. Suppose that $A.2$ is not satisfied and $\succeq$ is not strongly T-acyclic, but there exists an ex-post stable and efficient, and locally envy-free rule f. Recall that in the proof of $I.1^*$, it is shown that an ex-post constrained efficient and locally envy-free rule does not exist if there is a type-I cycle. So there exists a type-II* cycle or a weak cycle. It can be easily seen that a type-II* cycle also involves a weak cycle. Hence, it is sufficient to consider the case in which there are distinct $i, j, k \in N$ and $a, b \in A$ such that $q_a = q_b = 1$ and $i \succeq_a j \succ_a k \succeq_b i$. Consider the following preference profile.

$$R: \quad \begin{array}{ll} i: & b, a, \emptyset \\ j: & a, \emptyset \\ k: & a, b, \emptyset \end{array}$$

Similar to the proof of Theorem 2, it is easy to show the following. First, by ex-post stability and efficiency, $f_{ia}(R) = 0$. Second, by ex-post stability or local envy-freeness, if $f_{ib}(R) = 1$, then $f_{ka}(R) = 1$. But this implies $f_{ja}(R) = 0$ and contradicts ex-post stability. Hence, $f_{ib}(R) < 1$.

Since $f_{i\emptyset}(R) > 0$, by ex-post stability (when $i \succ_a j$) or local envy-freeness (when $i \sim_a j$), $f_{ja}(R) \neq 1$. Let λ be a lottery that induces $f(R)$ such that each $\mu \in \mathscr{S}(\lambda)$ is

stable and efficient. Then there exists $\mu \in \mathscr{S}(\lambda)$ with $\mu(j) = \emptyset$ and $\mu(i) \neq a$. As μ respects priorities, $\mu(k) \neq a$. So a is wasted and a contradiction is reached. □

In the rest of this section, we give examples to show that the impossibility results can not be further strengthened.

Independence of the axioms in I.1. First, the following example shows that local envy-freeness is crucial (for all the impossibility results).

Example B.1 ($\succeq$ *is not T-acyclic, and there exists a strategy-proof, ex-post stable and efficient, and strongly symmetric rule*). By $A.1$, there exist distinct $1, 2, 3 \in N$ and $a, b \in A$ such that $|N| \geq q_a + q_b + 1$. Let $\sigma \in \mathscr{O}$. The priority structure is as follows: $1 \sim_a 3 \succ_a 2$, $1 \succ_c 2 \sim_c 3$ for all $c \in A \setminus \{a\}$, $N \setminus \{1, 2, 3\} \subseteq SU(\succeq_c, 1)$ for all $c \in A$, and for all $c \in A$ and $i, j \in N \setminus \{1, 2, 3\}$, $i \succ_c j$ whenever $\sigma(i) < \sigma(j)$. $\succeq$ is not T-acyclic since there is a type-I cycle in which $1 \succ_b 2 \sim_b 3$ and $3 \succeq_a 1$. Construct a rule f as follows. First, the agents in $N \setminus \{1, 2, 3\}$ choose their best available choices sequentially according to the ordering σ, and leave the problem with their assignments. Then, consider the reduced problem.

(1) a is exhausted or there are at least two copies of a available. Then the priority structure in the reduced problem is strongly T-acyclic. So apply DA-STB to it.

(2) There is exactly one copy of a available.

  (a) Agent 1's first choice is $c \in \bar{A} \setminus \{a\}$. 1 is assigned c and leaves the problem with her assignment. Apply DA-STB to the further reduced problem.

  (b) Agent 1's first choice is a.

    (i) Agent 3's first choice is a. Each of 1 and 3 is assigned 0.5 of a. After a is exhausted, the three agents consume the probability shares of their best available choices sequentially in the order of $1, 2, 3$.[23]

    (ii) Agent 3's first choice is not a. 1 is assigned a and leaves the problem with a. Apply DA-STB to the further reduced problem.

[23] For example, their preferences are as follows. $R_1 : a, b$; $R_2 : a, c$; $R_3 : a, c, b$. Each object has unit capacity. Then agent 1 consumes 0.5 of b, agent 2 consumes 1 of c and agent 3 consumes 0.5 of b. This is essentially an *eating algorithm* from Bogomolnaia and Moulin (2001). Hence in this case the resulting allocation M is *sd-efficient*, i.e., there does not exist a different allocation M' such that M'_i first-order stochastically dominates M_i for all $i \in N$. It can also be verified that M is *ex-ante stable* (Roth et al., 1993), which implies that there do not exist $i, j \in N$ and $c \in A$ such that $i \succ_c j$, $M_{jc} > 0$ and $\sum_{d \in U(R_i, c)} M_{id} < 1$. It follows from sd-efficiency and ex-ante stability that M must be ex-post stable and efficient.

It is not difficult to verify that f is strategy-proof, ex-post stable and efficient, and strongly symmetric. The key in this construction is that (i) and (ii) guarantee that agent 3 cannot manipulate. Notice that in case (b) agent 3 can only get a share of a by reporting it as her first choice, but then she has to consume another object after agent 2. □

Second, strategy-proofness cannot be dropped.

Example B.2 ($\succeq$ *is not T-acyclic, and there exists an ex-post constrained efficient, symmetric and locally envy-free rule*). The setup is the same as Example B.1 except the relative priority rankings among 1,2 and 3: let $1 \sim_a 2 \sim_a 3$ and $1 \succ_c 2 \succ_c 3$ for all $c \in A \setminus \{a\}$. Then $\succeq$ is not T-acyclic. Construct a rule f as follows. First, as in the previous example, let the agents in $N \setminus \{1, 2, 3\}$ choose their best available choices sequentially according to σ, and leave the problem with their assignments. Then, consider the reduced problem.

- a is exhausted or there are at least two copies of a available. Then the priority structure in the reduced problem is T-acyclic. So apply DA-STB to it.
- There is exactly one copy of a available.
  - (1) Agent 1's first choice is $c \in \bar{A} \setminus \{a\}$. 1 is assigned c and leaves the problem with her assignment. Apply DA-STB to the further reduced problem.
  - (2) Agent 1's first choice is a, and there are in total $n > 1$ agents whose first choices are a. Then each of these n agents is assigned $\frac{1}{n}$ of a. After a is exhausted, the three agents consume the probability shares of their best available choices sequentially in the order of $1, 2, 3$.
  - (3) Agent 1 is the only agent whose first choice is a.
    - (a) Agent 2 and agent 3 do not have the same first choice, or their common first choice has more than one copies available. Then each agent is assigned her first choice.
    - (b) Agent 2 and agent 3 have the same first choice with exactly one copy available.
      - (i) Agent 3's second choice is not a. Then 1 and 2 are assigned their first choices, and 3 is assigned her second choice.
      - (ii) Agent 3's second choice is a. Then each of 1 and 3 is assigned 0.5 of a. After a is exhausted, the three agents consume their best available choices sequentially in the order of $1, 2, 3$.

The symmetry and local envy-freeness of f can be easily verified. f is ex-post stable

and efficient except one case in (ii): if 1's second choice coincides with 3's first choice, then the allocation chosen by f is only ex-post constrained efficient. $\square$

Third, under the assumption $A.2$, symmetry cannot be dropped. In the following example, we additionally show that strategy-proofness cannot be dropped from $I.3$ when $A.2$ is satisfied.

Example B.3 (*$A.2$ is satisfied and $\succeq$ is not T-acyclic. There exists a strategy-proof, ex-post stable and efficient, and locally envy-free rule, and there exists an ex-post stable and efficient, locally envy-free and symmetric rule*). By $A.2$, we can find distinct $a, b \in A$ such that $q_a \geq 2, q_a \leq q_c$ for all $c \in A$ with $q_c \geq 2$, $q_b \leq q_c$ for all $c \in A \setminus \{a\}$, and $|N| \geq q_a + q_b + 1$. Let $1 \in N$ and $N' \subseteq N$ such that $1 \notin N'$ and $|N'| = q_a + q_b$. Consider the following priority structure. For any $c \in A \setminus \{a\}$, $i, j \in N'$ and $k \in N \setminus (N' \cup \{1\})$, we have $1 \sim_a i \sim_a j \succ_a k$ and $1 \succ_c i \sim_c j \succ_c k$. Moreover, given $\hat{\sigma} \in \mathcal{O}$, let $i \succ_c j$ if $c \in A$, $i, j \in N \setminus (N' \cup \{1\})$ and $\hat{\sigma}(i) < \hat{\sigma}(j)$. It can be easily seen that $\succeq$ is not T-acyclic. Construct a rule f as follows. If agent 1's first choice is not a, let 1 be assigned her first choice and leave with her assignment. Then apply DA-STB to the reduced problem. If agent 1's first choice is a, apply a modified random serial dictatorship in which each ordering $\sigma \in \mathcal{O}$ satisfying the following conditions is picked with equal probability: $\sigma(1) \leq |N'|, \sigma(i) \leq |N'| + 1$ for all $i \in N'$, and $\sigma(i) < \sigma(j)$ for all $i, j \in N \setminus (N' \cup \{1\})$ with $\hat{\sigma}(i) < \hat{\sigma}(j)$. Denote the set of such orderings as $\mathcal{O}'$.

Let $f^{SD}(\sigma, \cdot)$ be the serial dictatorship rule with respect to an ordering σ. For any R and $\sigma \in \mathcal{O}'$, if agent 1's first two choices are a and $c \in \bar{A} \setminus \{a\}$, then $f_1^{SD}(\sigma, R) R_1 c$, since $q_a + q_c \geq |N'| \geq \sigma(1)$. Given this, it can be easily seen that f is strategy-proof, and ex-post stable and efficient. It remains to show local envy-freeness. Suppose that for some R and $i \in N'$, $f_{1a}(R) = 1$, $f_{ia}(R) = 0$ and $f_{ic}(R) > 0$ for some $c \in \bar{A}$ with $a P_i c$. Then there exists $\sigma \in \mathcal{O}'$ such that $f_1^{SD}(\sigma, R) = a$ and $f_i^{SD}(\sigma, R) = c$. As $q_a \geq 2$, there is some $j \in N'$ with $f_j^{SD}(\sigma, R) = a$. Clearly $\sigma(1) < \sigma(i)$ and $\sigma(j) < \sigma(i)$. Construct two new orderings based on σ: let σ' be the ordering obtained by moving 1 down such that 1 is just above i;[24] let σ'' be the ordering obtained (from σ) by moving j down such that j is just below i. Then $\sigma', \sigma'' \in \mathcal{O}'$. First, we argue that $f_k^{SD}(\sigma, R) R_k a$ for all $k \in N$ with $\sigma(k) < \sigma(i)$. Assume this is not true. Then let $k \neq i$ be the first agent whose assignment is worse than a. Clearly $\sigma(1) < \sigma(k) < \sigma(i)$. Consider any agent l above k with $l \neq 1$.

[24] Formally, $\sigma'(1) = \sigma(i) - 1$, $\sigma'(k) = \sigma(k) - 1$ if $\sigma(1) < \sigma(k) < \sigma(i)$, and $\sigma'(k) = \sigma(k)$ if $\sigma(k) < \sigma(1)$ or $\sigma(k) \geq \sigma(i)$.

Since l chooses an object weakly better than a under σ, l must choose the same object under σ'. It follows that $f_k^{SD}(\sigma', R) = a$ and $f_1^{SD}(\sigma', R) \neq a$, contradicting to $f_{1a}(R) = 1$. Next, consider any agent l above i with $l \neq j$. Since l chooses an object weakly better than a under σ, l chooses the same object under σ''. This implies $f_i^{SD}(\sigma'', R) = a$, contradicting to $f_{ia}(R) = 0$. Local envy-freeness with respect to two agents in N' can be shown in a similar (and easier) way.

We construct another rule f' as follows.

(1) Agent 1's first choice is not a, or, her first choice is a and her second choice is not $\emptyset$. Let $f' = f$.

(2) Agent 1's first choice is a and her second choice is $\emptyset$. Apply a modified random serial dictatorship in which each ordering $\sigma \in \mathcal{O}$ satisfying the following conditions is picked with equal probability: $\sigma(i) \leq |N'| + 1$ for all $i \in N' \cup \{1\}$, and $\sigma(i) < \sigma(j)$ for all $i, j \in N \setminus (N' \cup \{1\})$ with $\hat{\sigma}(i) < \hat{\sigma}(j)$.

It can be easily checked that f' is ex-post stable and efficient, locally envy-free and symmetric. Notice that in case (1) the symmetry constraints only apply to those agents in N', and in case (2) agent 1's priority cannot be violated by any agent in N'. □

Finally, in $I.1$ ex-post constrained efficiency cannot be weakened to ex-post stability, as DA-STB is always strategy-proof, ex-post stable, symmetric and locally envy-free.

Independence of the axioms in I.2. The *fractional deferred acceptance mechanism* from Kesten and Ünver (2015) is ex-post stable, symmetric at the top and locally envy-free. Random serial dictatorship is strategy-proof, symmetric at the top and locally envy-free. DA-STB is strategy-proof, ex-post stable and locally envy-free. Finally, local envy-freeness cannot be dropped, as shown in Example B.1.

Independence of the axioms in I.3. DA-STB is strategy-proof, ex-post stable, symmetric and locally envy-free. Random serial dictatorship is strategy-proof, ex-post efficient, symmetric and locally envy-free. Local envy-freeness cannot be dropped as shown in Example B.1. Finally, when $A.2$ is satisfied, strategy-proofness or symmetry cannot be dropped as shown in Example B.3.

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