Ex-ante fair random allocation*

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Abstract

When allocating indivisible objects, agents might have equal priority rights for some objects. A common practice is to break the ties using a lottery and randomize over deterministic allocation mechanisms. Such randomizations usually lead to unfairness and inefficiency ex-ante. We propose a concept of ex-ante fairness and show the existence of an agent-optimal ex-ante fair solution. Ex-ante fair random allocations are generated using "allocation by division", a new method of generating random allocations from deterministic allocation mechanisms. Some important results from the two-sided matching theory and the recent random assignment literature are unified and extended. The set of ex-ante fair random allocations forms a complete lattice under first-order stochastic dominance relations. The agent-optimal ex-ante fair mechanism includes both the deferred acceptance algorithm and the probabilistic serial mechanism as special cases.

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1 Introduction

We consider the problem of allocating indivisible objects, such as houses, tasks, organs or public high school seats, to agents when monetary transfers are not possible. While agents have preferences over heterogeneous objects, each object often has its own priority ranking over the agents. For instance, in school choice programs, public high schools usually have priorities over students based on exam scores or "walk zone and sibling rules".[1] Moreover, coarse priority rankings are common: a group of students may be given equal priority at a certain school. Pure indivisible object allocation problems without specific priorities, i.e., *house allocation* (Hylland and Zeckhauser, 1979), also include a (degenerate) priority structure: all the agents have equal claim and are ranked equally for each object. In this paper, we study fair and efficient solutions to such *priority-augmented* allocation problems, allowing for general and, possibly, weak priority structures.

Given the possible ties in priorities, random allocations are necessary to restore fairness. The most common way of generating random allocations, both in theory and in practice, is to first break the ties in priorities using a randomly selected ordering of agents, then apply a deterministic allocation mechanism. Examples include random serial dictatorship (RSD) in house allocation (Abdulkadiroğlu and Sönmez, 1998)[2], top trading cycles mechanism in house allocation with existing tenants (Abdulkadiroğlu and Sönmez, 1999, Sönmez and Ünver, 2005) and deferred acceptance mechanism in school choice (Abdulkadiroğlu and Sönmez, 2003). Randomizing over deterministic allocation mechanisms generally preserves strategy-proofness of the deterministic allocation mechanisms, but the resulting allocations might not be (constrained) efficient or fair from the ex-ante perspective (Bogomolnaia and Moulin, 2001, Erdil and Ergin, 2008, Kesten and Ünver, 2015). We propose a new and normatively appealing fairness concept, *ex-ante fairness*, which is generally not satisfied by existing random allocation mechanisms. Ex-ante constrained

[1] When schools use the walk zone and sibling rules, a student's priority at a school is determined by two criteria: whether the student is living within the walk zone of this school and whether the student has a sibling studying at this school.

[2] RSD selects a random allocation by picking an ordering of the agents from the uniform distribution and letting the agents choose their favorite available object sequentially according to this ordering. If we want to allocate one object between two agents with equal claim, tossing a coin is the simplest form of RSD. This results in a random allocation in which each agent receives this object with 0.5 probability. Abdulkadiroğlu and Sönmez (1998) show that RSD is equivalent to randomizing over core mechanisms.

efficiency can be achieved: we provide a mechanism that selects the unique ex-ante fair random allocation which first-order stochastically dominates every other ex-ante fair random allocation. Instead of the usual randomization method, ex-ante fair allocations are generated using *allocation by division*, a new way of extending deterministic allocation mechanisms to the probabilistic setting.

Ex-ante fairness is defined as the combination of two axioms: *ex-ante stability* and *ordinal fairness*. The original stability concept for deterministic allocations (Gale and Shapley, 1962, Abdulkadiroğlu and Sönmez, 2003) says that an object should not be given to an agent with a lower priority when it is desired by an agent with a greater claim to it. Ex-ante stability is the natural generalization of this concept to the probabilistic setting. It requires that if one agent is ranked higher than another agent for some object, then it is not possible for the second agent to receive this object with a positive probability, while the first agent receives a worse object with a positive probability.[3] Ordinal fairness, on the other hand, requires that if some agents are ranked equally by an object, the allocation of this object among them should try to equalize each one's probability of obtaining a weakly better object. This is a natural adaption of the original ordinal fairness axiom defined by Hashimoto et al. (2014) in house allocation to our context. They show that ordinal fairness characterizes *probabilistic serial mechanism* (PS) proposed by Bogomolnaia and Moulin (2001).[4] In PS, the agents consume the probability shares of their best available objects simultaneously at the unit rate. This new solution to house allocation has superior efficiency and fairness properties from the ex-ante perspective compared to RSD.

The introduction of PS leads to a growing literature on random assignment. While ordinal fairness is the defining feature of PS, ex-ante stability corresponds to the central concept in classical two-sided matching theory starting from Gale and Shapley (1962). Consequently, by studying ex-ante fairness, results in this paper unify and extend important solutions and insights from these two strands of matching literature.

Our first set of results establish that the set of ex-ante fair random allocations is well-behaved. In particular, it is a complete and distributive lattice under the first-order stochastic dominance relation of the agents. Hence there exists a unique

[3]Ex-ante stability was initially proposed in Roth et al. (1993) and also studied in Manjunath (2015) and Kesten and Ünver (2015).

[4] If there are unequal numbers of objects and agents, then a mild condition, non-wastefulness, is also required to characterize PS.

agent-optimal ex-ante fair random allocation. We also establish a generalized version of rural hospital theorem (Roth, 1986): each agent or object's probability of being assigned is the same across all the ex-ante fair random allocations. In some context, objects are not entirely passive and have intrinsic preferences that are aligned with their priorities.[5] In this case, agents and objects have opposite interests over ex-ante fair random allocations; each ex-ante fair random allocation is stochastic-dominance efficient for the two sides of the market. All of these results generalize the key insights regarding stable deterministic allocations from the classical two-sided matching theory.[6] They do not hold for stable deterministic allocations when priorities are weak. One general question in two-sided matching theory is to what extent the properties of stable allocations depend on the specific model assumptions. It has long been believed that strict orderings on both sides of the market is an indispensable assumption. Hence our main contribution to this line of research is to establish that the lattice structure, as well as many other properties of stable allocations, holds under weak priorities if we consider random allocations and define stability properly.[7]

We construct *generalized deferred acceptance mechanism* (GDA) which selects the unique agent-optimal ex-ante fair random allocation. When priorities are strict, GDA reduces to the *deferred acceptance algorithm* of Gale and Shapley (1962), which selects the (deterministic) agent-optimal stable allocation. GDA reduces to PS when priorities are degenerate. Hence our first contribution to the random assignment literature is to provide the first generalization of PS to priority-augmented allocation problems. More importantly, we reinterpret the idea behind the original PS solution as a new method of generating random allocations through deterministic allocation mechanisms: *allocation by division*.

The basic idea behind this method is as follows. We first divide agents and objects into equal number (or measure) of parts. Then, we define strict priorities over the parts of each agent and treat parts of agents in the same priority group symmetrically. Finally, applying a stable deterministic allocation mechanism to this divided

[5]For instance, in school choice, if public high schools' priorities are determined by students' exam scores, then it is likely that the schools' preferences are consistent with priorities.

[6]See Roth and Sotomayor (1992) for a survey on the classical results in two-sided matching.

[7]It is worth noting that two important results from the two-sided matching theory cannot be generalized into our framework: stability is equivalent to the core and the agent-optimal stable mechanism is strategy-proof. We discuss such incentive issues in more details in Section 5.

problem generates a fair random allocation for the original problem.[8] Given a general priority structure, the nature of the problem requires that each agent or object has to be divided into a continuum of parts. A stable deterministic allocation in the continuum matching problem gives an ex-ante fair random allocation for the original problem. Applying the deferred acceptance algorithm to the continuum problem leads to our GDA. However, for simple priority structures it is sufficient to divide agents and objects into finite number of parts. In fact, the previous generalizations of PS can be obtained by applying some form of serial dictatorship to finitely divided problems. In particular, the generalization of PS to house allocation with weak preferences (Katta and Sethuraman, 2006) can be obtained from the serial dictatorships of Svensson (1994). The generalization of PS to allocation problems with initial property rights (Yılmaz, 2010) can be obtained from individually rational serial dictatorships. Two generalizations of PS to house allocation with multi-unit demands (Kojima, 2009, Heo, 2014) can be obtained from two forms of serial dictatorships under multi-unit demands (Pápai, 2000, 2001, Klaus and Miyagawa, 2001, Ehlers and Klaus, 2003, Bogolmonaia et al., 2014). Finally, Budish et al. (2013) generalize PS to accommodate various constraint structures, such as controlled choice requirements in school choice. This version of PS can also be obtained from a constrained sequential allocation procedure.

Randomizing over deterministic mechanisms is a commonly observed practice to generate fair random allocations. When allocating one single object between two agents, tossing a coin is the simplest form of RSD. An alternative common solution is rotation or time-sharing: each agent has this object for half of the time. Such a method can be considered as a simple application of allocation by division. By formalizing this idea of allocation by division, it can be seen that all of the PS solutions, as well as our ex-ante fair solutions, have their foundations in deterministic mechanisms. As pointed out in Bogolmonaia et al. (2014), three usual ways of fair allocation in the presence of indivisibility are randomization, rotation and monetization. However, monetary transfers are often not appropriate or ethical in practical market design problems.

[8] As a simple and concrete example, consider the allocation of one single object between two agents with equal claim. We can divide each agent and the object into two parts, with the first parts of the two agents having an equal and higher priority than the second parts of them. Then, in any stable deterministic allocation, the first part of each agent receives one part of the object. This leads to a fair random allocation for the original problem, in which each agent receives this object with 0.5 probability.

1.1 Related literature

Our study is related to several strands of matching literature. First, after the initial seminal work of Gale and Shapley (1962), the key insights in the two-sided marriage problem have been generalized in different directions by several important studies, including Crawford and Knoer (1981), Kelso and Crawford (1982), Roth (1984), Roth et al. (1993), Alkan and Gale (2003) and Hatfield and Milgrom (2005). We complement this line of research by extending the theory to problems with one-sided weak orderings. Both Roth et al. (1993) and Alkan and Gale (2003) establish the lattice structure of stable random matchings, but the former assumes strict orderings and the latter takes a revealed preference approach.

The random assignment literature had been very small until the introduction of PS in the seminal work of Bogomolnaia and Moulin (2001). Many studies contribute to a better understanding of this intuitive new allocation mechanism that possesses superior fairness and efficiency properties. In addition to axiomatic characterizations of PS given by Bogomolnaia and Heo (2012) and Hashimoto et al. (2014), Kesten (2009) shows that PS can be viewed as a form of RSD or top trading cycles mechanism,[9] Che and Kojima (2010) show that PS and RSD are asymptotically equivalent in large markets, and recently Bogomolnaia (2015) provides a new and welfarist interpretation of PS. We contribute to this literature by reinterpreting the PS solutions as a method of allocation by division.

Priority-augmented allocation has been studied extensively in the context of school choice problems, starting from Abdulkadiroğlu and Sönmez (2003). Given weak priorities, one common solution is deferred acceptance algorithm with a fixed tie-breaking rule.[10] Such a mechanism is ex-post stable, but might not be fair from the ex-ante perspective. Kesten and Ünver (2015) is the first study on designing random allocation mechanisms that are fair or stable ex-ante in this context. Our ex-ante fairness is similar to their *strong ex-ante stability* which consists of ex-ante stability and another condition called *no ex-ante discrimination*. They construct *fractional deferred acceptance algorithm* (FDA) which is agent-optimal strongly ex-ante stable. Our GDA closely resembles FDA, but there is no logical relation between the two.

[9] Although the interpretation is different, Kesten (2009) also touches on the idea of allocation by division. His results imply that if we infinitely divide the standard house allocation problem under strict preferences, then serial dictatorships will converge to PS.

[10] Deferred acceptance algorithm with a fixed tiebreaking rule is currently used in many school choice programs in the U.S. See Abdulkadiroğlu et al. (2005a) and Abdulkadiroğlu et al. (2005b).

Finally, there are several recent studies on continuum matching markets, for instance, Abdulkadiroğlu et al. (2015), Azevedo and Hatfield (2015), Che et al. (2015) and Azevedo and Leshno (forthcoming). In these studies, continuum matching markets are used to model large markets and may help eliminate difficulties in finite matching problems. In this paper we do not focus on large markets. Instead, we use continuum matching models to obtain results regarding finite problems. However, the continuum matching models do help us eliminate the technical difficulties in applying the allocation by division method to problems with general priority structures.

In the next section we set up the model. Section 3 presents results regarding the structure of the set of ex-ante fair random allocations. Section 4 focuses on the generalized deferred acceptance mechanism. Section 5 discusses the method of allocation by division in general. Section 6 concludes. All the proofs are in the appendices.

2 Preliminaries

Let N be a finite set of agents and A a finite set of objects. Each agent $i \in N$ has a complete, transitive and antisymmetric **preference relation** R_i on $A \cup \{i\}$, where P_i denotes its asymmetric component. $R = (R_i)_{i \in N}$ is a **preference profile**. Each object $a \in A$ has a complete and transitive **priority ordering** $\succeq_a$ on $N \cup \{a\}$, where $\succ_a$ and $\sim_a$ denote its asymmetric and symmetric components, respectively. $\succeq = (\succeq_a)_{a \in A}$ is a **priority structure**. For simplicity, assume that each pair of agent and object is "mutually acceptable": for any $i \in N, a \in A$, $aP_i i$ and $i \succ_a a$.[11] A **problem** is summarized as $e = (N, A, R, \succeq)$.

Given $e = (N, A, R, \succeq)$, a **random allocation**, or simply an allocation, is a $|N| \times |A|$ matrix M with $M_{ia} \geq 0$, $\sum_{b \in A} M_{ib} \leq 1$, and $\sum_{j \in N} M_{ja} \leq 1$ for all $i \in N$ and $a \in A$. For $i \in N$, $M_i = (M_{ia})_{a \in A \cup \{i\}}$ denotes the lottery obtained by i, with $M_{ii} = 1 - \sum_{a \in A} M_{ia}$. M_a and M_{aa} are defined analogously for each $a \in A$. M is a **deterministic allocation** if $M_{ia} \in \{0, 1\}$ for all $i \in N$ and $a \in A$. By the *Birkhoff-von Neumann Theorem* (Birkhoff, 1946, von Neumann, 1953), every random allocation can be represented as a lottery over deterministic allocations.[12]

[11] All the results still hold in the setting where "being self-assigned" is not necessarily an agent's worst option, and each object can rank some agent below "being assigned nothing".

[12] See Budish et al. (2013) for a maximal generalization of this theorem to a broader class of alloca-

A random allocation M is **non-wasteful** if $M_{aa} > 0$ and $M_{ib} > 0$ imply bR_ia for any $i \in N, a \in A$ and $b \in A \cup \{i\}$. A deterministic allocation M is **stable** if it is non-wasteful and there is *no justified-envy*, i.e., there do not exist $i, j \in N, a \in A$ and $b \in A \cup \{i\}$ such that $i \succ_a j, M_{ib} = M_{ja} = 1$ and aP_ib. An agent i can compare some lotteries M_i and M'_i using the first-order stochastic dominance relation R_i^{sd}: $M_i R_i^{sd} M'_i$ if $\sum_{bR_ia} M_{ib} \geq \sum_{bR_ia} M'_{ib}$ for all $a \in A$. $\succeq_a^{sd}$ is defined analogously for each $a \in A$. A random allocation M is **sd-efficient** if there does not exist $M' \neq M$ with $M'_i R_i^{sd} M_i$ for all $i \in N$. A deterministic allocation M is **efficient** if it is sd-efficient.

A **mechanism** is a function f that assigns a random allocation to each problem. f is said to satisfy some property mentioned above if $f(e)$ satisfies this property for all e. f is **strategy-proof** if for any $e = (N, A, R, \succeq)$, $i \in N$ and $e' = (N, A, R' = (R'_i, R_{-i}), \succeq)$, $f_i(e) R_i^{sd} f_i(e')$.

3 Ex-ante fairness

Fix a problem $e = (N, A, R, \succeq)$ in this section. First, in a fair allocation the assignment of each object should always accommodate the demand of higher ranked agents first.

Definition 1 A random allocation M is **ex-ante stable** if (i) it is non-wasteful, and (ii) there do not exist $i, j \in N, a \in A$ and $b \in A \cup \{i\}$ such that $i \succ_a j, M_{ja} > 0, M_{ib} > 0$ and aP_ib.

Ex-ante stability is first proposed in Roth et al. (1993), and further studied in Manjunath (2015) and Kesten and Ünver (2015). Condition (ii) is essentially a probabilistic version of the no justified-envy condition. It requires that a random allocation respects the priority structure from the ex-ante prospective when agents compare their lotteries. An ex-ante stable random allocation can only be represented as lotteries over stable deterministic allocations.

Ex-ante stability does not put many restrictions on a random allocation if the priority structure is very coarse. Next, in light of the possible ties in priorities, we impose the following fairness condition.

Definition 2 A random allocation M is **ordinally fair** if for any $i, j \in N$ and $a \in A$ with $i \sim_a j$, $M_{ia} > 0$ implies $\sum_{bR_ja} M_{jb} \geq \sum_{bR_ia} M_{ib}$.

tion problems.

This concept is first introduced in Hashimoto et al. (2014) for house allocation problems, where every pair of agents is ranked equally by each object. Together with non-wastefulness, it characterizes *probabilistic serial mechanism* (PS) (Bogomolnaia and Moulin, 2001). Two standard fairness axioms in house allocation problems are *equal treatment of equals* and *sd-envy-freeness*. The former requires that any two agents with the same preferences should be assigned the same lottery, while the latter says that each agent's lottery should first-order stochastically dominate any other agent's lottery. These two axioms are not appropriate in our context since generally, agents may not have equal claims for every object. Thus fairness has to be defined "locally" at each object. Ordinal fairness requires that when two agents are ranked equally by some object a, the one who has a lower probability of being assigned a strictly better object should get a higher probability share of a. In other words, the allocation of the probability shares of an object among agents in the same priority class should try to equalize each of these agents' "surplus" at this object.

These two axioms constitute the central concept in this paper.

Definition 3 A random allocation M is **ex-ante fair** if it is ex-ante stable and ordinally fair.

If we restrict attention to strict priorities and deterministic allocations, then ex-ante fairness is equivalent to stability. An ex-ante fair random allocation always exists, but we postpone the construction of an ex-ante fair mechanism to the next section. Denote the set of ex-ante fair allocations as $\mathcal{E}$. In the rest of this section we discuss properties of ex-ante fair allocations.

First, since a stable and efficient deterministic allocation may not exist (Roth, 1982, Abdulkadiroğlu and Sönmez, 2003), ex-ante fairness and sd-efficiency are generally not compatible. In some context, objects have intrinsic preferences over the agents. If such preferences are consistent with the priorities, then any ex-ante fair allocation is "sd-efficient" for the two sides of the market.

Proposition 1 *For any $M \in \mathcal{E}$, there does not exist a random allocation $M' \neq M$ such that $M'_i R^{sd}_i M_i$ for all $i \in N$ and $M'_a \succeq^{sd}_a M_a$ for all $a \in A$.*

This generalizes the result that any stable deterministic allocation is efficient for the two sides of the market when priorities are strict. To further study and understand ex-ante fairness, we rely crucially on a continuum matching market that corresponds to our finite allocation problem.

3.1 A continuum matching market

Imagine that each agent or object is a *type*, and there is a continuum of copies of each type. For each $o \in N \cup A$, there is a bijection, also denoted as o, from $[0,1]$ to all the copies of type o. We take a new and natural deterministic approach to the random allocation problem.[13] Any random allocation of A to N in the original problem can be represented by a deterministic matching between the atomless objects $\{a(x)\}_{a\in A, x\in[0,1]}$ and the atomless agents $\{i(x)\}_{i\in N, x\in[0,1]}$. Let $\mathcal{C}$ be the collection of all the closed intervals in $[0,1]$. For our purpose, it is sufficient to consider how "intervals of objects", $\{a(F)\}_{a\in A, F\in\mathcal{C}}$, are matched with "intervals of agents", $\{i(F)\}_{i\in N, F\in\mathcal{C}}$.[14]

To complete the description of the continuum matching market, we extend preferences and priorities to the intervals. Given $F = [x,y] \in \mathcal{C}$ and $F' = [x',y'] \in \mathcal{C}$, denote $F < F'$ if $y \leq x'$. For each $a(F)$, define $\succ_{a(F)}$ as follows. Given any (possibly non-distinct) $i_1, i_2 \in N \cup \{a\}$ and $F_1, F_2 \in \mathcal{C}$, $i_1(F_1) \succ_{a(F)} i_2(F_2)$ if $i_1 \succ_a i_2$, or, $i_1 \sim_a i_2$ and $F_1 < F_2$. Thus for each type, atomless agents closer to zero are assumed to be higher ranked. Each $P_{i(F)}$ is defined analogously.

Generally there are multiple matchings of intervals that represent a random allocation. We want to translate the ex-ante fairness of a random allocation to the stability of some deterministic matching in the continuum market. Thus some restrictions have to be imposed on the matchings of intervals. Before formally defining a matching of intervals and its stability, we use the following simple example to illustrate the ideas behind them.

Example 1 Suppose $N = \{i,j\}$ and $A = \{a,b\}$. Preferences are given by aP_ib and aP_jb. Priorities are given by $i \sim_a j$ and $i \succ_b j$. The unique ex-ante fair allocation is M:

$$M_{ia} = M_{ib} = M_{ja} = M_{jb} = 0.5$$

Suppose that some "stable" matching of intervals μ represents M. First, it must be the case that $i([0,0.5])$ is matched with some intervals of a, since $i([0,0.5])$ is ranked higher than $i([0.5,1])$ for a, and a is preferred to b by every atomless agent of type i. Similarly, $j([0,0.5])$ is also matched with some intervals of a. Then which intervals

[13] This is in fact an application of the allocation by division method. We discuss this method in general in Section 5.

[14] In fact, we do not even need to specify the measure of the copies of each type. The important construction here is the bijection from $[0,1]$ to the copies of each type.

of a are assigned to $i([0,0.5])$ and $j([0,0.5])$ respectively? Intuitively, since $i \sim_a j$, the atomless objects $a([0,1])$ should be symmetrically distributed to $i([0,0.5])$ and $j([0,0.5])$ to respect the priorities. So we let

$$\mu\{i([0,0.5])\} = \mu\{j([0,0.5])\} = 0.5a([0,1])$$

and for any interval $[x,y] \subseteq [0,0.5]$,

$$\mu\{i([x,y])\} = \mu\{j([x,y])\} = 0.5a([2x,2y])$$

It can also be easily seen that $i \succ_b j$ requires that

$$\mu\{i([0.5,1])\} = b([0,0.5]), \mu\{j([0.5,1])\} = b([0.5,1])$$

We define a few more notations. A *partition* of $[0,1]$ is a finite set of real numbers $Q \subseteq [0,1]$, with $\{0,1\} \subseteq Q$. If $Q = \{x_1, x_2, ..., x_k\}$ and $x_1 < x_2 < ... < x_k$, let $\mathcal{F}(Q) = \{[x_1,x_2],[x_2,x_3],...,[x_{k-1},x_k]\}$. Given $F = [x,y]$, let $|F| = y - x$. Formally, an **interval matching** is a function μ defined on the domain $\{o(F) : o \in N \cup A, F \in \mathcal{C}\}$, which satisfies the following properties.

A.1 (Partitions of agents) For each $i \in N$, there is a partition Q_i of $[0,1]$ such that for any $F \in \mathcal{F}(Q_i)$, $\mu(i(F)) = \{(\lambda, a(G))\}$ for some $a \in A \cup \{i\}$, $G \in \mathcal{C}$ and $\lambda \in (0,1]$ with $|F| = \lambda|G|$, where $F = G$ if $a = i$.

A.2 (Partitions of objects) For each $a \in A$, there is a partition Q_a such that for any $F \in \mathcal{F}(Q_a)$, either $\mu(a(F)) = \{i_1(G), ..., i_k(G)\}$ for some k distinct agents with $i_1 \sim_a i_2 \sim_a ... \sim_a i_k$, and $G \in \mathcal{C}$ with $|F| = k|G|$, or $\mu(a(F)) = \{a(F)\}$.

A.3 (Mutual matching) For any $i \in N$ and $F \in \mathcal{F}(Q_i)$, if $\mu(i(F)) = \{(\lambda, a(G))\}$ and $a \in A$, then $G \in \mathcal{F}(Q_a)$ and $i(F) \in \mu(a(G))$. For any $a \in A$ and $F \in \mathcal{F}(Q_a)$, if $\mu(a(F)) = \{i_1(G), ..., i_k(G)\}$ and $i_1 \neq a$, then $G \in Q_{i_l}$ and $\mu(i_l(G)) = \left\{(\frac{1}{k}, a(F))\right\}$ for each $l \in \{1, ..., k\}$.

A.4 (Uniformity) Given $F, G \in \mathcal{C}$, define the linear function $f : F \to G$. Let $F' \subseteq F$ and $G' = f(F')$. For each $i \in N$, if $F \in \mathcal{F}(Q_i)$ and $\mu(i(F)) = \{(\lambda, a(G))\}$, then $\mu(i(F')) = \{(\lambda, a(G'))\}$. For each $a \in A$, if $F \in \mathcal{F}(Q_a)$ and $\mu(a(F)) = \{i_1(G), ..., i_k(G)\}$, then $\mu(a(F')) = \{i_1(G'), ..., i_k(G')\}$.

A.5 (General intervals) For any $o \in N \cup A$ and $F = [x, y] \in \mathcal{C}$, construct a set of intervals $C = \{G \in \mathcal{F}(Q_o \cup \{x, y\}) : |G \cap F| > 0\}$. Then $\mu(o(F)) = \cup_{G \in C} \mu(i(G))$.

A.6 (Monotonicity) Let $F, G \in \mathcal{C}$ and $F < G$. For any $i \in N$, $\mu(i(F)) P_{i(F)} \mu(i(G))$. For any $a \in A$, $\mu(a(F)) \succ_{a(F)} \mu(a(G))$.[15]

An interval matching μ **induces** a unique well-defined random allocation M:

$$\forall i \in N, a \in A, M_{ia} = \sum_{F \in \mathcal{F}(Q_i): i(F) \in \mu(a([0,1]))} |F|$$

For any random allocation, there always exists an interval matching that induces it.[16] The details are given in Appendix B.

Definition 4 An interval matching μ is **F-stable** if there do not exist $i \in N, a \in A$ and $F, G \in \mathcal{C}$ such that $a(G) P_{i(F)} \mu(i(F))$ and $i(F) \succ_{a(G)} \mu(a(G))$.

Lemma 1 *Suppose that μ induces M, then M is ex-ante fair if and only if μ is F-stable.*

3.2 The structure of the set of ex-ante fair random allocations

Given two interval matchings μ and μ', we can define the set of atmoless agents who prefer μ' to μ, as well as the set of atomless objects that are matched with higher ranked agents at μ':[17]

$$\forall i \in N, \mathcal{B}_i(\mu') = cl \left\{ \cup_{F \in \mathcal{C}: \mu'(i(F)) P_{i(F)} \mu(i(F))} F \right\}$$

$$\forall a \in A, \mathcal{B}_a(\mu') = cl \left\{ \cup_{F \in \mathcal{C}: \mu'(a(F)) \succ_{a(F)} \mu(a(F))} F \right\}$$

The following result generalizes the decomposition lemma of Knuth (1976) in standard two-sided matching problems. It says that at any two F-stable matchings μ

[15] For simplicity, we abuse the notations slightly. Given $\mu_1 = \{(\lambda_1, a_1(F_1)), ..., (\lambda_k, a_k(F_k))\}$ and $\mu_2 = \left\{(\lambda'_1, b_1(G_1)), ..., (\lambda'_{k'}, b_{k'}(G_{k'}))\right\}$, denote $\mu_1 P_{i(F)} \mu_2$ if for any $l \in \{1, ..., k\}$ and $l' \in \{1, ..., k'\}$, $a_l(F_l) P_{i(F)} b_{l'}(G_{l'})$. Denote $a(F) P_{i(F)} \mu_1$ if for any $l \in \{1, ..., k\}$, $a(F) P_{i(F)} a_l(F_l)$. Each $\succ_{a(F)}$ is similarly extended.

[16] There could exist multiple interval matchings that induce one random allocation, but they are essentially equivalent: the differences are only due to different specifications of the partitions.

[17] Since each set defined here is the union of uncountably many closed intervals, we take the closure to make sure that it is closed.

and μ', the atomless agents who prefer μ' are matched with those atomless objects assigned higher ranked agents at μ.

Lemma 2 *Suppose that μ and μ' are F-stable. For any $i \in N$ and $F \in \mathcal{C}$, if $F \subseteq \mathcal{B}_i(\mu')$, then $\mu(i(F)), \mu'(i(F)) \subseteq \{(\lambda, a(G)) : a \in A, G \subseteq \mathcal{B}_a(\mu)\}$. For any $a \in A$ and $F \in \mathcal{C}$, if $F \subseteq \mathcal{B}_a(\mu)$, then $\mu(a(F)), \mu'(a(F)) \subseteq \{i(G) : i \in N, G \subseteq \mathcal{B}_i(\mu')\}$.*

An immediate result from this decomposition lemma is that an agent's probability of being assigned some object, as well as an object's probability of being assigned to some agent, is constant across all the ex-ante fair random allocations. This generalizes the *rural hospital theorem* (Roth, 1986).

Corollary 1 For any $M, M' \in \mathcal{E}$, $i \in N$ and $a \in A$, $\sum_{b\in A} M_{ib} = \sum_{b\in A} M'_{ib}$, and $\sum_{j\in N} M_{ja} = \sum_{j\in N} M'_{ja}$.

We establish the lattice structure of $\mathcal{E}$ using Lemma 2. For any M and M', denote $MR_N^{sd}M'$ if $M_iR_i^{sd}M'_i$ for all $i \in N$. $\succeq_A^{sd}$ is defined analogously. Given $M, M' \in \mathcal{E}$, define matrices $M \vee_N M'$ and $M \wedge_N M'$ as follows:

$$(M \vee_N M')_{ia} = max\left\{\sum_{bR_ia} M_{ib}, \sum_{bR_ia} M'_{ib}\right\} - max\left\{\sum_{bP_ia} M_{ib}, \sum_{bP_ia} M'_{ib}\right\}$$

$$(M \wedge_N M')_{ia} = min\left\{\sum_{bR_ia} M_{ib}, \sum_{bR_ia} M'_{ib}\right\} - min\left\{\sum_{bP_ia} M_{ib}, \sum_{bP_ia} M'_{ib}\right\}$$

If these matrices are well-defined random allocations, then they are the least upper bound (the join) and the greatest lower bound (the meet) of $\{M, M'\}$, respectively, under the antisymmetric relation R_N^{sd}.

Generally, $\succeq_A^{sd}$ is not a partial order due to the possible ties in priorities. But as shown in Proposition 2, it is a partial order on $\mathcal{E}$. So if some random allocation $\bar{M}$ satisfies

$$\sum_{j\sim_a i} \bar{M}_{ja} = max\left\{\sum_{j\succeq_a i} M_{ja}, \sum_{j\succeq_a i} M'_{ja}\right\} - max\left\{\sum_{j\succ_a i} M_{ja}, \sum_{j\succ_a i} M'_{ja}\right\}$$

for all $i \in N$ and $a \in A$, then $\bar{M}$ is the least upper bound of $\{M, M'\}$, under $\succeq_A^{sd}$. In this case, let $M \vee_A M' = \bar{M}$. Similarly, if some random allocation $\underline{M}$ satisfies

$$\sum_{j \sim_a i} \underline{M}_{ja} = min\left\{\sum_{j \succeq_a i} M_{ja}, \sum_{j \succeq_a i} M'_{ja}\right\} - min\left\{\sum_{j \succ_a i} M_{ja}, \sum_{j \succ_a i} M'_{ja}\right\}$$

for all $i \in N$ and $a \in A$, then $\underline{M}$ is the greatest lower bound of $\{M, M'\}$, under $\succeq_A^{sd}$. In this case, let $M \wedge_A M' = \underline{M}$.

A main result of this paper is that $\mathcal{E}$ is a lattice with respect to R_N^{sd} or $\succeq_A^{sd}$.

Proposition 2 *For any $M, M' \in \mathcal{E}$, we have $M \vee_N M', M \wedge_N M' \in \mathcal{E}$. Moreover, $\succeq_A^{sd}$ is antisymmetric on $\mathcal{E}$, $M \vee_A M' = M \wedge_N M'$ and $M \wedge_A M' = M \vee_N M'$.*

An immediate consequence from the lattice structure is that if objects have intrinsic preferences consistent with priority orderings, then agents and objects have conflicting interests regarding ex-ante fair allocations.

Corollary 2 *For any $M, M' \in \mathcal{E}$, $M R_N^{sd} M'$ if and only if $M' \succeq_A^{sd} M$.*

Different from the standard two-sided matching problem, $\mathcal{E}$ might be infinite. Given any subset $S \subseteq \mathcal{E}$, define the matrices $sup(S)$ and $inf(S)$ as follows:

$$[sup(S)]_{ia} = sup\left\{\sum_{bR_ia} M_{ib} : M \in S\right\} - sup\left\{\sum_{bP_ia} M_{ib} : M \in S\right\}$$

$$[inf(S)]_{ia} = inf\left\{\sum_{bR_ia} M_{ib} : M \in S\right\} - inf\left\{\sum_{bP_ia} M_{ib} : M \in S\right\}$$

If $sup(S)$ and $inf(S)$ are well-defined random allocations, then they are the least upper bound and the greatest lower bound of S under R_N^{sd} , respectively. The next result shows that the lattice $(\mathcal{E}, R_N^{sd})$ is complete, and distributive.

Proposition 3 *For any $S \subseteq \mathcal{E}$, $sup(S), inf(S) \in \mathcal{E}$; for any $M^1, M^2, M^3 \in \mathcal{E}$, $M^1 \wedge_N (M^2 \vee_N M^3) = (M^1 \wedge_N M^2) \vee_N (M^1 \wedge_N M^3)$, and $M^1 \vee_N (M^2 \wedge_N M^3) = (M^1 \vee_N M^2) \wedge_N (M^1 \vee_N M^3)$.*

In light of Corollary 2, the lattice $(\mathcal{E}, \succeq_A^{sd})$ is also complete and distributive. Completeness implies that there exists a unique **agent-optimal ex-ante fair random allocation**, which first-order stochastically dominates all the ex-ante fair random allocations. The main results in this section (Proposition 1, Lemma 2, Corollary 1, Proposition 2, Corollary 2 and Proposition 3) generalize the corresponding results in the standard two-sided matching problem. None of these results holds for stable deterministic allocations under weak priorities.

4 Agent-optimal ex-ante fair mechanism

The structure of the set of ex-ante fair random allocations is established by exploring the connection with F-stable interval matchings. A stable deterministic matching in the continuum matching market induces an ex-ante fair random allocation for the original problem. We can apply *deferred acceptance algorithm* of Gale and Shapley (1962) to the continuum matching market. This will lead to a generalized deferred acceptance procedure for the original problem, which selects the agent-optimal ex-ante fair allocation. It is sufficient to study intervals of agents in the continuum problem, so in the generalized deferred acceptance procedure for the original problem, we will have *fractions* of agents applying to objects.

Given some fractions of agents $\{i_1(F_1), i_2(F_2), ..., i_k(F_k)\}$ with $\sum_{l=1}^{k} |F_l| > 1$, clearly an object a can choose the "best" fractions of agents such that the sum of the measures of these fractions is 1, according to the priority relation $\succ_{a([0,1])}$ defined in Section 2.1. Given $e = (N, A, R, \succeq)$, we now define the generalized deferred acceptance procedure as follows.

Step 1 The whole unit of each agent applies to her favorite object. Each object tentatively accepts the best fractions from the applicants, such that the sum of the measures of these fractions does not exceed one, and rejects all the other fractions of applicants.

Step *k* If a fraction F of an agent i is rejected by some object in step $k-1$, then the fraction F of i applies to i's next best object. Each object chooses from the applying fractions as well as those tentatively accepted fractions , tentatively accepts the best fractions such that the sum does not exceed one, and rejects all the other fractions.

For each k, define the tentative assignment matrix $M^k(e)$ for the agents as follows. For any i and a, let $M^k_{ia}(e)$ be the sum of the measures of fractions of i that are either tentatively accepted by a at step $k-1$ (i.e., on the waiting list of a at step $k-1$), or applying to a at step k. $M^k(e)$ is generally not a well-defined random allocation. For instance, $M^1_{ia}(e) = 1$ for any agent i whose favorite object is a.

The described deferred acceptance procedure will generally not terminate in a finite number of steps. For any i, a and k, $M^k_{ia}(e) = \sum_{bR_ia} M^k_{ib}(e) - \sum_{bP_ia} M^k_{ib}(e)$. It can be easily seen that, due to the deferred acceptance procedure, both $\{\sum_{bR_ia} M^k_{ib}(e)\}_{k=1}^{\infty}$ and $\{\sum_{bP_ia} M^k_{ib}(e)\}_{k=1}^{\infty}$ are bounded decreasing sequences of real numbers. Hence $\{M^k_{ia}(e)\}_{k=1}^{\infty}$ is convergent. Let $lim_{k\to\infty}M^k(e)$ denote the limit of this sequence of matrices, i.e., for any i and a, $[lim_{k\to\infty}M^k(e)]_{ia} = lim_{k\to\infty}[M^k_{ia}(e)]$.

Proposition 4 *For any e, $lim_{k\to\infty}M^k(e)$ is a well-defined random allocation and it is agent-optimal ex-ante fair.*

Let $f^{GDA}(e) = lim_{k\to\infty}M^k(e)$. f^{GDA} is **generalized deferred acceptance mechanism (GDA)**. The simple and neat structure of the convergent sequence of tentative assignments $\{M^k(e)\}_{k=1}^{\infty}$ helps us easily establish the ex-ante fairness of $f^{GDA}(e)$. The agents become worse-off in each step: $M^k_i(e)R^{sd}_iM^{k+1}_i(e)$ for all k and i. In the proof we show that the tentative assignment at each step first-order stochastically dominates every ex-ante fair allocation for all the agents. Hence, in the limit, $f^{GDA}(e)$ is agent-optimal ex-ante fair.

GDA closely resembles the *fractional deferred acceptance algorithm* (FDA) in Kesten and Ünver (2015). In FDA, it is irrelevant that which fraction of an agent is applying, and an object tries to tentatively accept an equal fraction of each equally ranked applying agent. Instead of ex-ante fairness, they propose a notion of *strong ex-ante stability* which consists of ex-ante stability and *no ex-ante discrimination*. Then FDA selects the agent-optimal strongly ex-ante stable random allocation.[18]

GDA generalizes the original deferred acceptance algorithm from Gale and Shapley (1962). It also generalizes the probabilistic serial mechanism (PS) from Bogomolnaia and Moulin (2001). Given a problem $e = (N, A, R, \succeq)$, where all the agents are ranked equally by each object, PS selects a random allocation as follows.

Let $A_1 = A$ and $r_1(a) = 1$ for each $a \in A$. Given $i \in N$ and $A' \subseteq A$, let $B_i(A')$ denote the maximal element in A' according to R_i. At step k, $k \geq 1$, for each $a \in A_k$, define

$$\lambda_k(a) = \frac{r_k(a)}{|\{i \in N : B_i(A_k) = a\}|}$$

[18]An allocation M satisfies no ex-ante discrimination if there do not exist $i, j \in N$ and $a \in A$ with $i \sim_a j$ such that $M_{ia} < M_{ja}$, and $M_{ib} > 0$ for some $b \in A \cup \{i\}$ with aP_ib. This condition says that the allocation of an object among equally ranked agents should take into account of each one's probability of obtaining some strictly worse object. In contrast, ordinal fairness says that the allocation should consider each one's probability of obtaining a strictly better object.

where $\lambda_k(a) = +\infty$ if $\{i \in N : B_i(A_k) = a\} = \phi$, then

$$\lambda_k = min_{a \in A_k} \{\lambda_k(a)\}$$

Each $i \in N$ is assigned λ_k of $B_i(A_k)$ at step k. For each $a \in A$, $r_{k+1}(a) = r_k(a) - \lambda_k | \{i \in N : B_i(A_k) = a\} |$. Then $A_{k+1} = \{a \in A : r_{k+1}(a) > 0\}$. The process terminates at some step k' if $r_{k'+1}(a) = 0$ for all $a \in A$ or $\sum_{k=1}^{k'} \lambda_k = 1$.

Intuitively, all the agents are consuming, or "eating", their best available objects at the speed of one simultaneously. It can be easily seen that for this special priority structure, GDA is reduced to PS. Alternatively, this equivalence also follows from the fact that, in house allocation problems, ex-ante fairness is equivalent to the combination of ordinal fairness and non-wastefulness, and PS is characterized by these two axioms (Hashimoto et al., 2014).

In spite of the interesting ex-ante properties, GDA is not strategy-proof, which follows from the non-strategy-proofness of PS. In fact, GDA is not even weakly strategy-proof, i.e., an agent could potentially manipulate her preferences to obtain a strictly better lottery. In contrast, DA with fixed tie-breaking, a common mechanism in school choice problems, is strategy-proof and ex-post stable. But such a mechanism is not ex-ante stable (Kesten and Ünver, 2015), ordinally fair or sd-efficient for the two sides of the market (Bogomolnaia and Moulin, 2001). Its outcome can even be dominated within the set of ex-post stable allocations (Erdil and Ergin, 2008).

5 Allocation by division

The construction of ex-ante fair solutions is based on a new method of extending deterministic allocation mechanisms to random environment, which we call *allocation by division*. The general idea behind allocation by division is as follows. We first divide agents and objects into parts. Then we grant different priority rights to different parts of each agent and treat the parts of equally ranked agents symmetrically. Finally, applying a stable deterministic allocation mechanism to this divided problem will generate a fair random allocation for the original problem. To accommodate all the possible weak priority structures, in Section 3.1 we have to divide each agent or object into a continuum of parts.[19] A stable deterministic allocation in the continuum problem gives an ex-ante fair random allocation for the original problem. In Section 4, we see that applying the deferred acceptance algorithm to the continuum problem leads to GDA. However, when the priority structure is relatively simple, it is sufficient to divide agents and objects into a finite number of parts.

For simplicity, in the rest of this section assume $|N| = |A|$ in any problem $e = (N, A, R, \succeq)$. Given e, we define the corresponding *C-divided problem* $e^C = (N^C, A^C, R^C, \succeq^C)$, where C is a positive integer. All the agents and objects are divided into C parts: $N^C = \{i_k\}_{k\in\{1,...,C\},i\in N}$ and $A^C = \{a_k\}_{k\in\{1,...,C\},a\in A}$. For any $i_k \in N^C$ and $a_{k_1}, b_{k_2} \in A^C$, $a_{k_1} R^C_{i_k} b_{k_2}$ if aR_ib. For any $i_{k_1}, j_{k_2} \in N^C$ and $a_k \in A^C$, $i_{k_1} \succeq^C_{a_k} j_{k_2}$ if $i \succ_a j$, or, $i \sim_a j$ and $k_1 \leq k_2$. A deterministic allocation M^C for e^C will give a random allocation M for e: for any $i \in N$ and $a \in A$,

$$M_{ia} = \frac{1}{C} | \left\{ k \in \{1, 2, ..., C\} : M^C_{i_k a_{k'}} = 1 \, \textit{for some } k' \right\} |$$

We show that the previous generalizations of PS can be obtained by finitely dividing those allocation problems and applying corresponding deterministic allocation mechanisms.

First, Katta and Sethuraman (2006) propose an **extended PS** solution to house allocation problems with weak preferences. The extended PS is a correspondence, but it is *essentially single-valued* since each agent is indifferent between any two allocations selected by this correspondence. Let E^{HA} denote the class of house allocation problems with weak preferences. Given any $e = (N, A, R, \succeq) \in E^{HA}$, the extended PS is defined as follows.[20]

Let $A_0 = A$ and $s_0(i) = 0$ for all $i \in N$. Given $i \in N' \subseteq N$ and $A' \subseteq A$, let $B_i(A')$ denote the set of maximal elements in A' according to R_i, and $B_{N'}(A') = \cup_{i\in N'} B_i(A')$. At step k, $k \geq 1$, let N_k be the largest set of agents such that

$$N_k \in argmin_{N' \subseteq N, N' \neq \phi} \frac{|B_{N'}(A_{k-1})| - \sum_{i\in N'} s_{k-1}(i)}{|N'|}$$

[19] This is related to the fact that GDA is not a finite procedure. If finite division is sufficient to generate the agent-optimal ex-ante fair allocation, then GDA would be a finite procedure.

[20] Here we adopt the definition from Heo and Yılmaz (2015), who also characterize the extended PS using ordinal fairness and non-wastefulness. See Katta and Sethuraman (2006) for the original definition using networks.

Let

$$\lambda_k = \frac{|B_{N_k}(A_{k-1})| - \sum_{i \in N_k} s_{k-1}(i)}{|N_k|}$$

Let $A_k = A_{k-1} \setminus B_{N_k}(A_{k-1})$. Then the objects in $B_{N_k}(A_{k-1})$ are allocated to N_k at step k: each $i \in N_k$ is assigned the objects in $B_i(A_{k-1})$ with a probability of $\lambda_k + s_{k-1}(i)$. Let $s_k(i) = 0$ for each $i \in N_k$ and $s_k(i) = s_{k-1}(i) + \lambda_k$ for $i \in N \setminus N_k$. The algorithm terminates if for some k', $A_{k'} = \phi$. Denote the resulting set of allocations as $f^{EPS}(e)$.

In the extended PS, the agents are still consuming the objects at the unit rate, but to guarantee sd-efficiency, the "bottleneck sets" (N_k) have to be identified. A class of deterministic solutions to E^{HA}, **serial dictatorships**, are given by Svensson (1994). Given $e \in E^{HA}$ and an ordering σ of agents, where $\sigma : \{1, 2, ..., |N|\} \to N$, σ is a bijection, define the following sequence of choice sets:

$$A_{\sigma(1)} = A$$
$$A_{\sigma(k)} = \left\{ a \in A : \exists\, allocation\ M, \forall l < k, M_{\sigma(l)b} = 1, b \in B_{\sigma(l)}(A_{\sigma(l)})\ and\ M_{\sigma(k)a} = 1 \right\}$$

Then the serial dictatorship with respect to σ selects the set of deterministic allocations $\{M : \forall i \in N, M_{ia} = 1\ for\ some\ a \in B_i(A_i)\}$. Obviously a serial dictatorship is an essentially single-valued correspondence. A deterministic allocation is efficient if and only if it is selected by some serial dictatorship. Given $e \in E^{HA}$, serial dictatorships can be applied to e^C. To respect the priorities in e^C, we restrict attention to the set of orderings $\sigma(e^C)$, where for each $\sigma \in \sigma(e^C)$, $k_1 < k_2$ implies $\sigma(i_{k_1}) < \sigma(j_{k_2})$ for all $i, j \in N$. Then let $f^{SD}(C, e)$ denote the set of random allocations that are generated from applying the serial dictatorships with respect to each $\sigma \in \sigma(e^C)$ to the divided problem e^C.

Proposition 5 *For any $e = (N, A, R, \succeq) \in E^{HA}$, there exists C such that $f^{SD}(C, e)$ and $f^{EPS}(e)$ are essentially equivalent: for any $M \in f^{SD}(C, e)$ and $M' \in f^{EPS}(e)$, $MR_N^{sd}M'$ and $M'R_N^{sd}M$. In particular, $f^{SD}(((|N|!)^{|A|}, e)$ and $f^{EPS}(e)$ are essentially equivalent.*

Next, we consider allocation problems with initial property rights, or *house allocation with existing tenants* (Abdulkadiroğlu and Sönmez, 1999).[21] Initial property rights can be modeled as a special type of priority structures: the owner of an object

[21] A special case of such problems is *housing market* (Shapley and Scarf, 1974), where each object is initially owned by one agent.

is ranked strictly higher than all the other agents. Then the requirement of *individual rationality*, i.e., an agent's random allocation must first-order stochastically dominate the degenerate lottery of receiving her endowment, is captured by ex-ante stability. Formally, let E^{HET} denote the set of allocation problems with initial property rights. For each $e = (N, A, R, \succeq) \in E^{HET}$, there exist non-empty $N(e) \subseteq N$ and a one-to-one function $h_e : N(e) \to A$ such that (1) for each $a \in h_e(N(e))$, $h_e^{-1}(a) \succ_a i \sim_a j$ for all $i, j \in N \setminus \{h_e^{-1}(a)\}$, and (2) for each $a \notin h_e(N(e)), i \sim_a j$ for all $i, j \in N$.[22] For $i \in N' \subseteq N(e)$, denote $U_i = \{a \in A : aR_i h_e(i)\}$ and $U_{N'} = \cup_{i \in N'} U_i$. Yılmaz (2010) defines **individually rational PS**.[23] Under this mechanism, agents are still consuming their best available objects at the unit rate, but to satisfy the individual rationality constraints, for any $N' \subseteq N(e)$, the objects $U_{N'}$ have to be entitled to N'. Specifically, suppose that some $i \notin N'$ is also consuming some object in $U_{N'}$. If at some point, continuing to consume this object implies that some agent in N' has to consume an object worse than her endowment later, then i is asked to stop consuming anything in $U_{N'}$. Denote the individually rational PS as f^{IRPS}. Its formal definition is given in Appendix E.

Given $e \in E^{HET}$, the set of efficient and individually rational deterministic allocations is characterized by **individually rational serial dictatorships**, which are constructed similarly to the serial dictatorships of Svensson (1994). Given an ordering σ, consider the following choice sets:

$$A_{\sigma(1)} = \left\{ a \in A : \exists \textit{ stable deterministic } M, M_{\sigma(1)a} = 1 \right\}$$

$$A_{\sigma(k)} =$$
$$\left\{ a \in A : \exists \textit{ stable deterministic } M, M_{\sigma(k)a} = 1, \forall l < k, M_{\sigma(l)B_{\sigma(l)}(A_{\sigma(l)})} = 1 \right\}$$

Then the individually rational serial dictatorship selects the deterministic allocation M with $M_{iB_i(A_i)} = 1$ for all $i \in N$. Given the corresponding divided problem e^C, let $f^{IRSD}(C, e)$ denote the set of random allocations for e which are generated from

[22]Assume that preferences are strict. Yılmaz (2009) generalizes PS to the allocation problems with initial property rights and weak preferences, which incorporates the techniques from Katta and Sethuraman (2006) and Yılmaz (2010). Our results can be easily extended to this case, but we consider weak preferences and initial property rights separately to better understand the ideas behind each case.

[23]Individually rational PS is sd-efficient and satisfies an envy-free condition, but it is not ordinally fair. GDA can be applied to E^{HET}, but it is not sd-efficient. Generally (sd-) efficiency and (ex-ante) stability are not compatible, but for special priority structures efficient and stable solutions do exist.

applying the individually rational serial dictatorships with respect to each $\sigma \in \sigma(e^C)$ to the divided problem e^C.

Proposition 6 *For any $e = (N, A, R, \succeq) \in E^{HET}$, there exists C such that $f^{IRSD}(C, e)$ is single-valued and $f^{IRSD}(C, e) = f^{IRPS}(e)$. In particular, $f^{IRSD}(((|N|!)^{|A|^2}, e) = f^{IRPS}(e)$.*

Both the extended PS and the individually rational PS are defined based on the "eating process" in the original PS. Katta and Sethuraman (2006) transform the problem into a network flow problem and use the *max-flow min-cut theorem* to show that the extended PS is well-defined. On the other hand, Yılmaz (2010) uses the *supply-demand theorem* (Gale, 1957) to show that the individually rational PS is well-defined. f^{SD} and f^{IRSD} are always well-defined. In establishing their equivalence to the extended PS and the individually rational PS respectively, we simply invoke *Hall's theorem*. This is not surprising, since logically Hall's theorem is based on both the max-flow min-cut theorem and the supply-demand theorem.[24]

There are several other generalizations of PS in previous studies. We can establish similar results as Proposition 5 and 6. The techniques are largely the same, so we omit the formal treatment of these cases. First, in house allocation problems with multi-unit demands, there are two forms of serial dictatorships. An agent with a demand of q can pick the best q objects available when it is her turn to pick.[25] Applying this mechanism to some divided problem can generate the PS generalized (and characterized) by Heo (2014), in which an agent with a demand of q has an eating speed of q. In the second form of serial dictatorships, each agent picks only one object when it is her turn and there are multiple rounds of sequential allocations until everyone's demand is satisfied.[26] Applying this mechanism to some divided problem will generate the PS generalized by Kojima (2009), in which an agent with a demand of q is still eating at unit rate, but she can eat from $t = 0$ to $t = q$. Finally, Budish et al. (2013) generalizes PS to accommodate various constraint structures in practical allocation problems, such as controlled choice requirements

[24]Specifically, Hall's theorem can be proved from the max-flow min-cut theorem, and it is a special (and discrete) case of the supply-demand theorem.

[25]For studies on this mechanism, see Pápai (2000, 2001), Klaus and Miyagawa (2001), Ehlers and Klaus (2003), Bogolmonaia et al. (2014).

[26]See Bogolmonaia et al. (2014) for a discussion on this mechanism. While the first type of serial dictatorships is generally strategy-proof, the second type is not. Abdulkadiroğlu and Sönmez (1998) show that RSD is equivalent to the mechanism that selects the core from random endowments. Using the second type of serial dictatorships, Bogolmonaia et al. (2014) generalize this result to the case of multi-unit demands.

in school choice. Agents are still eating the best available objects at the unit rate, but the availability of objects depends on the constraint structure. This version of PS can also be obtained by applying a constrained sequential allocation procedure to finitely divided problems.

The PS solutions, as well as our ex-ante fair solutions, are broadly interpreted as random allocation mechanisms generated from deterministic allocation mechanisms using the allocation by division method. Compared to randomizing over deterministic allocation mechanisms, this class of mechanisms generally has superior efficiency and fairness properties from the ex-ante perspective, but cannot preserve strategy-proofness. This is mainly due to the fact that generally the ex-ante properties are not compatible with strategy-proofness, as suggested in previous impossibility results (Zhou, 1990, Bogomolnaia and Moulin, 2001, Katta and Sethuraman, 2006, Kojima, 2009, Heo, 2014, Kesten and Ünver, 2015, Han, 2016). Both methods are commonly observed in practice. For instance, when allocating one single object between two agents, we could either flip a coin and let one agent have this object (randomization), or let each agent have this object for half of the time (allocation by division). To apply the allocation by division method, we need to divide the agents and objects into some "correct" number of parts: allocating three time slots between two agents cannot be fair. When allocating one object among some set of agents N, to accommodate all the possible cases the object and the agents can be divided in to $|N|!$ parts. Proposition 5 shows that, to allocation more than one object, we can simply raise $|N|!$ to the power of the number of objects. Dividing the objects and agents into $(|N|!)^{|A|}$ parts and applying sequential allocation mechanisms can lead to the unique ordinally fair random allocation.[27]

6 Concluding remarks

In this paper, we propose a new fair solution to the allocation problems with weak priority structures, which encompasses important results from both the two-sided matching and the one-sided matching literature. Such a unification sheds some new lights on several interesting open questions in the matching literature, including the

[27] Although the number is generally very large, it is a finite number and works for all the possible preference profiles. In Proposition 6, the number is enlarged to $(|N|!)^{|A|^2}$. This is because that there are more steps when allocating the objects in the presence of the individual rationality constraints.

robustness of the properties of stable matchings to weak orderings, and the connection between PS and deterministic allocation mechanisms.

Throughout the paper we have restricted attention to the one-to-one matching context, but the results can be easily extended to allow multiple copies of each object, as in school choice. In this case, to establish the lattice structure and related results, we only need to modify the construction of the continuum matching market by mapping all the atomless objects of each type to a larger interval, according to the quota of each object. Similarly, in defining GDA, an object can tentatively accept fractions of agents up to its quota in each step.

Although weak priorities are allowed in the model, our main results cannot be generalized to the case of weak orderings on both sides of the market. When preferences are weak, the connection between stable deterministic allocations in the continuum problem and ex-ante fair random allocations in the original problem disappears, and the set of ex-ante fair random allocations is no longer a lattice. This is not surprising, since allocation by division is mainly a method to deal with equal priority rights.

For future research, there could be more interesting results in two-sided matching theory that can be generalized into our framework. The allocation by division method can potentially be applied to other classes of allocation or matching problems to generate fair random allocations. It is also interesting to explore the performance of this method, as well as its connection with the randomization method, in large markets.

Appendix A: Proofs of Proposition 1 and Lemma 1

Proof of Proposition 1. Suppose some M is ex-ante fair, but there exists $M' \neq M$ such that $M'_i R^{sd}_i M_i$ for all $i \in N$ and $M'_a \succeq^{sd}_a M_a$ for all $a \in A$. Consider some $i^* \in N$ with $M_{i^*} \neq M'_{i^*}$. $M'_{i^*} R^{sd}_{i^*} M_{i^*}$ implies that there exists some $a \in A$ such that $M'_{i^*a} > M_{i^*a}$. Let a^* be the maximal element in $\{a \in A : M'_{i^*a} > M_{i^*a}\}$ with respect to R_{i^*}. Then $M'_{i^*} R^{sd}_{i^*} M_{i^*}$ implies (i) for each $aP_{i^*}a^*$, $M'_{i^*a} = M_{i^*a}$, and (ii) there exists some $b \in A \cup \{i^*\}$ with $a^* P_{i^*} b$ and $M_{i^*b} > 0$. By (ii) and the ex-ante stability of M, we have $M_{a^*a^*} = 0$ and $j \succeq_{a^*} i^*$ for all j with $M_{ja^*} > 0$. Then consider a^*. $M'_{a^*} \succeq^{sd}_{a^*} M_{a^*}$ and $M'_{i^*a^*} > M_{i^*a^*}$ imply that we cannot have $j \succ_{a^*} i^*$ for all j with $M_{ja^*} > 0$. Therefore, there exists some j such that $j \sim_{a^*} i^*$, $M_{ja^*} > 0$ and $k \succeq_{a^*} j$ for all k with $M_{ka^*} > 0$. Then $M'_{a^*} \succeq^{sd}_{a^*} M_{a^*}$ and $M'_{i^*a^*} > M_{i^*a^*}$ imply that there exists some i^{**} such that $i^* \sim_{a^*} i^{**}$ and $M'_{i^{**}a^*} < M_{i^{**}a^*}$.

Now $M_{i^{**}} R^{sd}_{i^{**}} M'_{i^{**}}$ and $M'_{i^{**}a^*} < M_{i^{**}a^*}$ imply that there exists some $bP_{i^{**}}a^*$ and $M'_{i^{**}b} > M_{i^{**}b}$. Let a^{**} be the maximal element in $\{b \in A : M'_{i^{**}b} > M_{i^{**}b}\}$ with respect to $R_{i^{**}}$. Continue in this fashion, since N is finite, there exist a list of distinct agents $(i_1, i_2, ..., i_k)$ and a list of (possibly non-distinct) objects $(a_1, a_2, ..., a_k)$ such that for each $l \in \{1, 2, ..., k\}$, we have $a_l P_{i_l} a_{l-1}$, $M'_{i_l a_l} > M_{i_l a_l}$, $M'_{i_l a_{l-1}} < M_{i_l a_{l-1}}$, and $i_{l-1} \sim_{a_{l-1}} i_l$, where $a_0 = a_k$, $i_0 = i_k$. Then for each l, since $M_{i_l a_{l-1}} > 0$, and M is ordinally fair, we have the following inequalities:

$$\sum_{aR_{i_{l-1}}a_{l-1}} M_{i_{l-1}a} \geq \sum_{aR_{i_l}a_{l-1}} M_{i_l a} > \sum_{aR_{i_l}a_l} M_{i_l a}$$

This implies that

$$\sum_{aR_{i_k}a_k} M_{i_k a} > \sum_{aR_{i_1}a_1} M_{i_1 a} > \sum_{aR_{i_2}a_2} M_{i_2 a} > ... > \sum_{aR_{i_k}a_k} M_{i_k a}$$

Clearly a contradiction is reached. □

Proof of Lemma 1. *"if" part.* Suppose μ is F-stable. Assume M is not ex-ante stable, then there exist $i \in N$ and $a \in A$ such that $M_{ib} > 0$ for some $b \in A \cup \{i\}$ with $aP_i b$, and $M_{ja} > 0$ for some $j \in N \cup \{a\}$ with $i \succ_a j$. By $A.1$ there exist F, G such that $\mu(i(F)) = \{(\lambda, b(G))\}$ for some λ. By $A.2$ there exist $F' \in \mathcal{F}(Q_a)$ and G' such that $j(G') \in \mu(a(F'))$. Then $aP_i b$ implies $a(F')P_{i(F)}\mu(i(F))$. Since $k \sim_a j$ for any $k(G') \in \mu(a(F'))$ and $i \succ_a j$, we have $i(F) \succ_{a(F')} \mu(a(F'))$. This contradicts to the

F-stability of μ.

Now suppose M is not ordinally fair. There exist $i, j \in N, a \in A$ with $i \sim_a j$, $M_{ia} > 0$, and $\sum_{bR_ja} M_{jb} < \sum_{bR_ia} M_{ib}$. By A.1 and A.6, there exist ϵ_1, ϵ_2 such that $\sum_{bR_ja} M_{jb} < \epsilon_2 < \epsilon_1 < \sum_{bR_ia} M_{ib}$, $\epsilon_1 > \sum_{bP_ia} M_{ib}$ and $F_i = [\epsilon_1, \sum_{bR_ia} M_{ib}] \subseteq F$ for some $F \in \mathcal{F}(Q_i)$. Then by A.4 and A.6 there exist F_a and λ such that $\mu(i(F_i)) = \{(\lambda, a(F_a))\}$, where $F_a \subseteq F$ for some $F \in \mathcal{F}(Q_a)$. Let $F_j = [\sum_{bR_ja} M_{jb}, \epsilon_2]$, then $j(F_j) \succ_{a(F_a)} \mu(a(F_a))$ since $i(F_i) \in \mu(a(F_a))$, $i \sim_a j$ and $F_j < F_i$. Finally by A.6 and the construction of F_j, $a(F_a)P_{j(F_j)}\mu(j(F_j))$. This contradicts to the F-stability of μ.

"only if" part. Assume μ is not F-stable. There exist $i \in N, a \in A$ and $F_i, F_a \in \mathcal{C}$ such that $a(F_a)P_{i(F_i)}\mu(i(F_i))$ and $i(F_i) \succ_{a(F_a)} \mu(a(F_a))$. A.6 implies that there do not exist F and λ such that $(\lambda, a(F)) \in \mu(i(F_i))$, so there exists some $(\lambda, b(F_b)) \in \mu(i(F_i))$ with aP_ib. Similarly, by A.6, there does not exist any F such that $i(F) \in \mu(a(F_a))$. Then there are two cases to consider.

Case 1. There exist some F and j with $i \succ_a j$ such that $j(F) \in \mu(a(F_a))$. Then $M_{ja} > 0$. Since $M_{ib} > 0$, the ex-ante stability of M is violated.

Case 2. There exist some j and F_j such that $i \sim_a j$, $F_i < F_j$ and $j(F_j) \in \mu(a(F_a))$. Then $M_{ja} > 0$, $\sum_{cR_ja} M_{jc} \geq max(F_j)$. Since $(\lambda, b(F_b)) \in \mu(i(F_i))$ and aP_ib, it follows that $\sum_{cR_ia} M_{ic} \leq max(F_i)$. Thus $\sum_{cR_ia} M_{ic} < \sum_{cR_ja} M_{jc}$, contradicting to the ordinal fairness of M. $\square$

Appendix B: Any random allocation is induced by some interval matching

Given a random allocation M, we construct an interval matching μ that induces M. Fix $a \in A$. Let $\mathcal{D} = \left\{i(F) : i \in N, F \in \mathcal{C}, min(F) = \sum_{bP_ia} M_{ib}, max(F) = \sum_{bR_ia} M_{ib}\right\}$. Partition $\mathcal{D}$ into $\mathcal{D}^1, \mathcal{D}^2, ..., \mathcal{D}^k$ such that for any $m, n \in \{1, 2, ..., k\}$, if $m < n$ and $i(F), i'(F') \in D^m, j(G) \in D^n$ then $i \sim_a i', i \succ_a j$. Consider any $m \in \{1, 2, ..., k\}$. Suppose some $i(F) \in \mathcal{D}^m$. Let $x = \sum_{j \succ_a i} M_{ja}$ and $y = \sum_{j \succeq_a i} M_{ja}$. The following iterative procedure specifies how the objects in $a([x, y])$ are matched with the agents in $\mathcal{D}^m$ by μ.

Let $\mathcal{D}^m_0 = \mathcal{D}^m, x_0 = x$. Given $\mathcal{D}^m_l, x_l, l \geq 0$, define

$$d_l = min\left\{\cup_{F : \exists i, i(F) \in \mathcal{D}^m_l} F\right\}$$

$$d'_l = min\left\{\left\{\cup_{F:\exists i, i(F)\in\mathcal{D}_l^m, min(F)\neq d_l} F\right\}\cup\left\{\cup_{F:\exists i, i(F)\in\mathcal{D}_l^m, min(F)=d_l} max(F)\right\}\right\}$$

Let $N_{d_l} = \left\{i\in N : \exists F, i(F)\in\mathcal{D}_l^m, min(F) = d_l\right\}$. Then

$$x_{l+1} = x_l + |N_{d_l}|(d'_l - d_l)$$

$$\mathcal{D}_{l+1}^m =$$

$$\left\{i(F)\in\mathcal{D}_l^m : i\notin N_{d_l}\right\}\cup\left\{i(F) : i\in N_{d_l}, \exists G, i(G)\in\mathcal{D}_l^m, F = G\setminus[d_l, d'_l]\neq\phi\right\}$$

Now for each $i\in N_{d_l}$, let $d_l, d'_l\in Q_i$,

$$\mu\left\{i([d_l, d'_l])\right\} = \left\{\left(\frac{1}{|N_{d_l}|}, a([x_l, x_{l+1}])\right)\right\}$$

For a, let $x_l, x_{l+1}\in Q_a$,

$$\mu\left\{a([x_l, x_{l+1}])\right\} = \left\{i([d_l, d'_l]) : i\in N_{d_l}\right\}$$

This process terminates when for some $l', x_{l'} = y$. Repeat this process for each $m\in\{1, 2, ..., k\}$ and each $a\in A$. Then by this construction, μ satisfies $A.1$, $A.2$ and $A.3$. We can impose $A.4$ and $A.5$ on μ, then $A.6$ is also satisfied.

Appendix C: Proofs for Section 3.2

To facilitate the proofs in this section, we first explicitly construct the sets $\mathcal{B}_o(\mu)$, $\mathcal{B}_o(\mu')$ for each $o\in N\cup A$.

Consider any two interval matchings μ and μ', with partitions $\{Q_o\}_{o\in N\cup A}$ and $\{Q'_o\}_{o\in N\cup A}$ respectively. For each o, let $Q''_o = Q_o\cup Q'_o$. For each $i\in N$, construct a collection of intervals $\mathcal{I}_i(\mu')$ as follows:
(i) If $F\in\mathcal{F}(Q''_i)$ and $\mu'(i(F))P_{i(F)}\mu(i(F))$, then let $F\in\mathcal{I}_i(\mu')$.
(ii) If $F\in\mathcal{F}(Q''_i)$, $\mu(i(F)) = \{(\lambda, a(G))\}$, $\mu'(i(F)) = \{(\lambda', a(G'))\}$, $a\in A$, and $G\neq G'$, then consider the linear functions $f : F\to G$, $f' : F\to G'$. Let $F' = \{x\in F : f'(x)\leq f(x)\}\in\mathcal{I}_i(\mu')$ if $|F'|\neq 0$.

Let $\mathcal{I}_i(\mu\mu') = \left\{F\in\mathcal{F}(Q''_i) : \mu'(i(F)) = \mu(i(F))\right\}$. By the construction, $\mathcal{I}_i(\mu\mu')$, $\mathcal{I}_i(\mu')$ and $\mathcal{I}_i(\mu)$ form a partition of $[0, 1]$.

Similarly, for each $a\in A$, construct $\mathcal{I}_a(\mu')$ as follows:

(i) If $F \in \mathcal{F}(Q''_a)$ and $\mu'(a(F)) \succ_{a(F)} \mu(a(F))$, then let $F \in \mathcal{I}_a(\mu')$.
(ii) If $F \in \mathcal{F}(Q''_a)$, $\mu(a(F)) = \{i_1(G), ..., i_k(G)\}$, $\mu'(a(F)) = \{j_1(G'), ..., j_{k'}(G')\}$, $i_1 \in N$, $i_1 \sim_a j_1$, and $G \neq G'$, consider the linear functions $f : F \to G$, $f' : F \to G'$. Let $F' = \{x \in F : f'(x) \leq f(x)\} \in \mathcal{I}_a(\mu')$ if $|F'| \neq 0$.

Let $\mathcal{I}_a(\mu\mu') = \{F \in \mathcal{F}(Q''_a) : \mu'(a(F)) = \mu(a(F))\}$. Clearly $\mathcal{I}_a(\mu\mu')$, $\mathcal{I}_a(\mu')$ and $\mathcal{I}_a(\mu)$ are disjoint. It can be easily seen that for each $o \in N \cup A$

$$\mathcal{B}_o(\mu) = \cup_{F \in \mathcal{I}_o(\mu)} F, \mathcal{B}_o(\mu') = \cup_{F \in \mathcal{I}_o(\mu')} F$$

Proof of Lemma 2. We first show the following two claims.

Claim 1. For any $i \in N$ and F_i with $\mu'(i(F_i)) = \{(\lambda, a(F_a))\}$, if $\mu'(i(F_i)) P_{i(F_i)} \mu(i(F_i))$, then $\mu(a(F_a)) \succ_{a(F_a)} \mu'(a(F_a))$.

Proof of Claim 1. Assume to the contrary, it is not true that $\mu(a(F_a)) \succ_{a(F_a)} \mu'(a(F_a))$. Then there exists some $F'_a \subseteq F_a$ with $\mu(a(F'_a)) = \{j_1(F_j), ..., j_k(F_j)\}$, $j_1 \sim_a ... \sim_a j_k$, such that we do not have $j_1(F_j) \succ_{a(F'_a)} i(F_i)$. If $i \succ_a j_1$, then $i(F_i) \succ_{a(F'_a)} \mu(a(F'_a))$ and $a(F'_a) P_{i(F_i)} \mu(i(F_i))$, contradicting to the F-stability of μ. If $i \sim_a j_1$, since we do not have $j_1(F_j) \succ_{a(F'_a)} i(F_i)$, it is not the case that $F_j < F_i$. So there are intervals $F'_i \subseteq F_i, F'_j \subseteq F_j$ such that $F'_i < F'_j$. Then there exists some $F''_a \subseteq F'_a$ with $\mu(a(F''_a)) = \left\{j_1(F'_j), ..., j_k(F'_j)\right\}$. Hence $i(F'_i) \succ_{a(F''_a)} \mu(a(F''_a))$ and $a(F''_a) P_{i(F'_i)} \mu(i(F'_i))$, contradicting to the F-stability of μ. $\square$

Claim 2. For any $a \in A$ and F_a with $F_a \subseteq F$ for some $F \in \mathcal{F}(Q_a)$, if $\mu(a(F_a)) \succ_{a(F_a)} \mu'(a(F_a))$, then $\mu'(i(F_i)) P_{i(F_i)} \mu(i(F_i))$ for all $i(F_i) \in \mu(a(F_a))$.

Proof of Claim 2. Assume to the contrary, for some $i(F_i) \in \mu(a(F_a))$, it is not true that $\mu'(i(F_i)) P_{i(F_i)} \mu(i(F_i))$. Then there exists $F'_i \subseteq F_i$ with $\mu'(i(F'_i)) = \{(\lambda, b(F_b)\}$ such that we do not have $b(F_b) P_{i(F'_i)} a(F_a)$. If $a P_i b$ then $a(F_a) P_{i(F'_i)} \mu'(i(F'_i))$ and $i(F'_i) \succ_{a(F_a)} \mu'(a(F_a))$, contradicting to the F-stability of μ'. If $b = a$, then it must be the case that $|F_a \cap F_b| = 0$, since $\mu(a(F_a)) \succ_{a(F_a)} \mu'(a(F_a))$. Hence $F_b > F_a$, and we do not have $\mu(a(F_a)) \succ_{a(F_a)} \mu(a(F_b))$, contradicting to $A.6$. $\square$

To show the decomposition lemma, it is sufficient to show that for any i and $F \in \mathcal{I}_i(\mu')$, $\mu(i(F)), \mu'(i(F)) \subseteq \{(\lambda, a(G)) : a \in A, G \subseteq \mathcal{B}_a(\mu)\}$, and for any a and $F \in \mathcal{I}_a(\mu)$, $\mu(a(F)), \mu'(a(F)) \subseteq \{i(G) : i \in N, G \subseteq \mathcal{B}_i(\mu')\}$.

Consider any i and $F \in \mathcal{I}_i(\mu')$. If $\mu'(i(F))P_{i(F)}\mu(i(F))$, then clearly $\mu'(i(F)) \subseteq \{(\lambda, a(G)) : a \in A, G \subseteq \mathcal{B}_a(\mu)\}$ by Claim 1. If it is not true that $\mu'(i(F))P_{i(F)}\mu(i(F))$, then by the construction of $\mathcal{I}_i(\mu')$, $\mu(i(F)) = \{(\lambda, a(G))\}$, $\mu'(i(F)) = \{(\lambda', a(G'))\}$, $a \in A$, $G \neq G'$, and $f'(x) \leq f(x)$ for all $x \in F$, where $f : F \to G$, $f' : F \to G'$, f and f' are linear. Clearly, $f(x) = f'(x)$ only if $x = max(F)$ or $x = min(F)$. Without loss of generality, suppose $f'(x) < f(x)$ for all $x \in F \setminus \{max(F)\}$. Then by the linearity of f and f', for any $\epsilon > 0$, there exists $\delta > 0$ such that for all $H \subseteq [min(F), max(F) - \epsilon]$ with $|H| < \delta$, $f'(H) < f(H)$. For such H, $\mu'(i(H))P_{i(H)}\mu(i(H))$, so by Claim 1 $f'(H) \subseteq \mathcal{B}_a(\mu)$. It follows that for any $\epsilon > 0$, $f'(F_\epsilon = [min(F), max(F) - \epsilon]) \subseteq \mathcal{B}_a(\mu)$. As $\epsilon \to 0$, $f'(F_\epsilon) \to G'$. Thus $G' \subseteq \mathcal{B}_a(\mu)$ since $\mathcal{B}_a(\mu)$ is closed. This shows that for any i and $F \in \mathcal{I}_i(\mu')$, $\mu'(i(F)) \subseteq \{(\lambda, a(G)) : a \in A, G \subseteq \mathcal{B}_a(\mu)\}$. Hence $\sum_{a \in A} \sum_{F \in \mathcal{I}_a(\mu)} |F| \geq \sum_{i \in N} \sum_{F \in \mathcal{I}_i(\mu')} |F|$.

By a similar argument, it can be shown that for any a and $F \in \mathcal{I}_a(\mu)$, $\mu(a(F)) \subseteq \{i(G) : i \in N, G \subseteq \mathcal{B}_i(\mu')\}$, which implies $\sum_{a \in A} \sum_{F \in \mathcal{I}_a(\mu)} |F| \leq \sum_{i \in N} \sum_{F \in \mathcal{I}_i(\mu')} |F|$. Finally we have $\sum_{a \in A} \sum_{F \in \mathcal{I}_a(\mu)} |F| = \sum_{i \in N} \sum_{F \in \mathcal{I}_i(\mu')} |F|$. It follows that for any i and $F \in \mathcal{I}_i(\mu')$, $\mu(i(F)), \mu'(i(F)) \subseteq \{(\lambda, a(G)) : a \in A, G \subseteq \mathcal{B}_a(\mu)\}$, and for any a and $F \in \mathcal{I}_a(\mu)$, $\mu(a(F)), \mu'(a(F)) \subseteq \{i(G) : i \in N, G \subseteq \mathcal{B}_i(\mu')\}$. $\square$

Proof of Proposition 2 Let $M, M' \in \mathcal{E}$. Suppose μ induces M and μ' induces M'. By Lemma 1 both μ and μ' are F-stable. We construct another interval matching in the following recursive way. For any $i \in N$ and $a \in A$, let $\mathcal{F}_a^0 = \mathcal{F}_i^0 = \phi$, and $\mathcal{I}_i^0 = \mathcal{I}_i(\mu) \cup \mathcal{I}_i(\mu') \cup \mathcal{I}_i(\mu, \mu')$. Let $\sigma : N \to \{1, 2, ..., |N|\}$, σ is a bijection. In general, at the k^{th} step, $k \geq 1$,

- If $\mathcal{I}_i^{k-1} \neq \phi$ and $\sigma(i) \leq \sigma(j)$ for any j with $\mathcal{I}_j^{k-1} \neq \phi$, let $i^k = i$.
- If $F \in \mathcal{I}_{i^k}^{k-1}$ and $F < G$ for any $G \in \mathcal{I}_{i^k}^{k-1}$ with $G \neq F$, let $F^k = F$.
- $\mu^k(i^k(F^k)) = \mu(i^k(F^k))$ if $F^k \subseteq \mathcal{B}_{i^k}(\mu)$, $\mu^k(i^k(F^k)) = \mu'(i^k(F^k))$ otherwise.
- $\mathcal{I}_{i^k}^k = \mathcal{I}_{i^k}^{k-1} \setminus \{F^k\}$, and $\mathcal{F}_{i^k}^k = \mathcal{F}_{i^k}^{k-1} \cup \{F^k\}$.
- If $\mu^k(i^k(F^k)) = \{(\lambda, a^k(G^k))\}$ for some $a^k \in A$ and G^k, let $\mathcal{F}_{a^k}^k = \mathcal{F}_{a^k}^{k-1} \cup \{G^k\}$.
- $\mu^k(a^k(G^k)) = \mu(a^k(G^k))$ if $\mu^k(i^k(F^k)) = \mu(i^k(F^k))$. $\mu^k(a^k(G^k)) = \mu'(a^k(G^k))$ if $\mu^k(i^k(F^k)) = \mu'(i^k(F^k))$.
- For each $i(F^k) \in \mu^k(a^k(G^k))$, let $\mu^k(i(F^k)) = \mu^k(i^k(F^k))$, $\mathcal{I}_i^k = \cup_{F \in \mathcal{I}_i^{k-1}} \{F \setminus F^k\}$, and $\mathcal{F}_i^k = \mathcal{F}_i^{k-1} \cup \{F^k\}$.
- For all the other i and a, let $\mathcal{I}_i^k = \mathcal{I}_i^{k-1}$, $\mathcal{F}_i^k = \mathcal{F}_i^{k-1}$ and $\mathcal{F}_a^k = \mathcal{F}_a^{k-1}$.

Let n be the smallest integer such that $\mathcal{I}_i^n = \phi$ for all $i \in N$, then the process terminates at step n. To see that this process of sequential allocation is well-defined, we invoke the decomposition lemma to show that each interval of agents or objects is assigned at most once.

First, at any step k, if $\mu^k(i^k(F^k)) = \{(\lambda, a^k(G^k))\}$, then $|G^k \cap F| = 0$ for any $F \in \mathcal{F}_{a^k}^{k-1}$. Suppose this is not true. Without loss of generality assume $\mu^k(i^k(F^k)) = \mu'(i^k(F^k))$, hence $\mu^k(a^k(G^k)) = \mu'(a^k(G^k))$. Then there exist $\bar{F^k} \subseteq F^k$ and $\bar{G^k} \subseteq G^k$ such that $\mu'(i^k(\bar{F^k})) = \left\{(\lambda, a^k(\bar{G^k}))\right\}$, and for some $k' < k$, $a^k = a^{k'}$, $\bar{G^k} \subseteq G^{k'}$. By Lemma 2, $\bar{G^k} \subseteq \mathcal{B}_{a^k}(\mu)$ or $\bar{G^k} \subseteq \cup_{F \in \mathcal{I}_{a^k}(\mu,\mu')} F$, thus by the construction, $G^{k'} \subseteq \mathcal{B}_{a^{k'}}(\mu)$ or $G^{k'} \subseteq \cup_{F \in \mathcal{I}_{a^{k'}}(\mu,\mu')} F$. By Lemma 2 again, $F^{k'} \subseteq \mathcal{B}_{i^{k'}}(\mu')$ or $F^{k'} \subseteq \cup_{F \in \mathcal{I}_{i^{k'}}(\mu,\mu')} F$. Hence $\mu^{k'}(a^{k'}(G^{k'})) = \mu'(a^{k'}(G^{k'}))$. This implies that $\bar{F^k} \subseteq \mathcal{F}_{i^k}^{k'} \setminus \mathcal{F}_{i^k}^{k'-1}$, contradicting to $F^k \in \mathcal{I}_{i^k}^{k-1}$.

Second, at any step k, if $i \neq i^k$, $i(F^k) \in \mu^k(a^k(G^k))$, then $|F^k \cap F| = 0$ for any $F \in \mathcal{F}_i^{k-1}$. Suppose this is not true. Without loss of generality assume $\mu^k(i^k(F^k)) = \mu'(i^k(F^k))$. Then there exist $i \neq i^k$, $\bar{F^k} \subseteq F^k$ and $\bar{G^k} \subseteq G^k$ such that $i(\bar{F^k}) \in \mu'(a^k(\bar{G^k}))$, and for some $k' < k$, $\bar{F^k} \subseteq \mathcal{F}_i^{k'} \setminus \mathcal{F}_i^{k'-1}$. By Lemma 2, $\mu^k(i^k(F^k)) = \mu'(i^k(F^k))$ implies $\bar{F^k} \subseteq \mathcal{B}_i(\mu')$ or $\bar{F^k} \subseteq \cup_{F \in \mathcal{I}_i(\mu,\mu')} F$. Hence by the construction, $\mathcal{F}_i^{k'} \setminus \mathcal{F}_i^{k'-1} \subseteq \mathcal{B}_i(\mu')$ or $\mathcal{F}_i^{k'} \setminus \mathcal{F}_i^{k'-1} \subseteq \cup_{F \in \mathcal{I}_i(\mu,\mu')} F$. Then $\mu^{k'}(i(\mathcal{F}_i^{k'} \setminus \mathcal{F}_i^{k'-1})) = \mu'(i(\mathcal{F}_i^{k'} \setminus \mathcal{F}_i^{k'-1}))$. It follows that $\bar{G^k} \subseteq G^{k'}$, contradicting to the statement shown in the last paragraph.

Given that each μ^k is well-defined, for any $o \in N \cup A$ and $F \in \mathcal{F}_o^k \setminus \mathcal{F}_o^{k-1}$, let $\mu^*(o(F)) = \mu^k(o(F))$. For each $i \in N$, let Q_i be the partition such that $\mathcal{F}(Q_i) = \mathcal{F}_i^n$. For each $a \in A$, let Q_a be the minimal partition such that $\mathcal{F}_a^n \subseteq \mathcal{F}(Q_a)$. For any a and $F \in \mathcal{F}(Q_a) \setminus \mathcal{F}_a^n$, let $\mu^*(a(F)) = \{a(F)\}$. Then μ^*, with respect to $\{Q_a\}_{a \in N \cup A}$, satisfies $A.1$, $A.2$ and $A.3$. We can impose $A.4$ and $A.5$ on μ^*. Thus it remains to show that $A.6$ (monotonicity) is satisfied. Consider any $i \in N$, $F, G \in \mathcal{F}(Q_i)$ and $F < G$. If $\mu^*(i(F)) = \mu(i(F))$ and $\mu^*(i(G)) = \mu(i(G))$, or $\mu^*(i(F)) = \mu'(i(F))$ and $\mu^*(i(G)) = \mu'(i(G))$, then $\mu^*(i(F)) P_{i(F)} \mu^*(i(G))$ by the monotonicity of μ and μ'. If $\mu^*(i(F)) = \mu(i(F)) \neq \mu'(i(F))$ and $\mu^*(i(G)) = \mu'(i(G))$ (the case in which $\mu^*(i(F)) = \mu'(i(F)) \neq \mu(i(F))$ and $\mu^*(i(G)) = \mu(i(G))$ is symmetric), then $F \subseteq \mathcal{B}_i(\mu)$. Combined with the fact that $\mu'(i(F)) P_{i(F)} \mu'(i(G))$, it is obvious that $\mu(i(F)) P_{i(F)} \mu'(i(G))$, i.e., $\mu^*(i(F)) P_{i(F)} \mu^*(i(G))$. By a similar argument, it can be shown that for any $a \in A$, $F, G \in \mathcal{F}(Q_a)$ and $F < G$, $\mu^*(a(F)) \succ_{a(F)} \mu^*(a(G))$.

So far it has been shown that μ^* is a well-defined interval matching, now we show that μ^* is F-stable. Suppose that this is not the case. Then there exist $i \in N, a \in$

A and F_i, F_a such that $a(F_a)P_{i(F_i)}\mu^*(i(F_i))$ and $i(F_i) \succ_{a(F_a)} \mu^*(a(F_a))$. Pick some $G_i \subseteq F_i$ and $G_a \subseteq F_a$ such that $G_i \subseteq F \in \mathcal{F}(Q_i)$ for some F and $G_a \subseteq F' \in \mathcal{F}(Q_a)$ for some F'. Clearly we still have $a(G_a)P_{i(G_i)}\mu^*(i(G_i))$ and $i(G_i) \succ_{a(G_a)} \mu^*(a(G_a))$. By the construction of μ^* we have $\mu^*(a(G_a)) = \mu(a(G_a))$ or $\mu^*(a(G_a)) = \mu'(a(G_a))$. Without loss of generality suppose $\mu^*(a(G_a)) = \mu(a(G_a))$, so $i(G_i) \succ_{a(G_a)} \mu(a(G_a))$. Then $a(G_a)P_{i(G_i)}\mu^*(i(G_i))$ implies $a(G_a)P_{i(G_i)}\mu(i(G_i))$, contradicting to the F-stability of μ.

It can be easily seen that μ^* induces $M \vee_N M'$. So $M \vee_N M' \in \mathcal{E}$ by Lemma 1. $M \wedge_A M' = M \vee_N M'$ also follows directly from the construction of μ^*. Now we show that $\succeq_A^{sd}$ is antisymmetric on $\mathcal{E}$. Suppose this is not the case, then there exist $M^1, M^2 \in \mathcal{E}$, $M^1 \neq M^2$, such that $M^1 \succeq_A^{sd} M^2$ and $M^2 \succeq_A^{sd} M^1$. Then $(M^1 \vee_N M^2)R_N^{sd}M^k$, $(M^1 \vee_N M^2) \succeq_A^{sd} M^k$, $k = 1, 2$. Since $M^1 \neq M^2$, we have $M^1 \vee_N M^2 \neq M^k$ for some k. Hence M^k is not sd-efficient for the two sides of the market, contradicting to Proposition 1. Finally, $M \wedge_N M' \in \mathcal{E}$ and $M \vee_A M' = M \wedge_N M'$ can be shown by a set of symmetric arguments. $\square$

Proof of Proposition 3 Consider any $S \subseteq \mathcal{E}$ and *sup(S)*. Clearly $[sup(S)]_{ia} \in [0, 1]$ for all $i \in N, a \in A$. Given any i and a, since

$$\sum_{bR_ia}[sup(S)]_{ib} = sup\left\{\sum_{bR_ia} M_{ib} : M \in S\right\}$$

it follows that $\sum_{b\in A}[sup(S)]_{ib} \leq 1$. To show that $sup(S)$ is a well-defined random allocation, it only remains to show that $\sum_{i\in N}[sup(S)]_{ia} \leq 1$ for all $a \in A$. Suppose this is not true for some $a \in A$. Then for each $i \in N$, there exists $M^i \in S$ such that

$$\sum_{bR_ia} M^i_{ib} > sup\left\{\sum_{bR_ia} M_{ib} : M \in S\right\} - \frac{1}{|N|}\left\{\sum_{j\in N}[sup(S)]_{ja} - 1]\right\}$$

Let $S' = \{M^i\}_{i\in N} \subseteq S$. Then we have for each i

$$
\begin{aligned}
[sup(S')]_{ia} =& sup\left\{\sum_{bR_ia} M_{ib} : M \in S'\right\} - sup\left\{\sum_{bP_ia} M_{ib} : M \in S'\right\} \\
>& sup\left\{\sum_{bR_ia} M_{ib} : M \in S\right\} - \frac{1}{|N|}\left\{\sum_{j\in N}[sup(S)]_{ja} - 1]\right\} - sup\left\{\sum_{bP_ia} M_{ib} : M \in S\right\} \\
=& [sup(S)]_{ia} - \frac{1}{|N|}\left\{\sum_{j\in N}[sup(S)]_{ja} - 1]\right\}
\end{aligned}
$$

Summing over N, we have

$$\sum_{i\in N}[sup(S')]_{ia} > 1$$

which contradicts to Proposition 3, since S' is finite.

Now we show that $sup(S) \in \mathcal{E}$. Suppose that $sup(S)$ is not ordinally fair. Then there exist $i, j \in N, a \in A$ such that $i \sim_a j, [sup(S)]_{ia} > 0$, and $\sum_{bR_ja}[sup(S)]_{jb} < \sum_{bR_ia}[sup(S)]_{ib}$. Pick some ϵ such that

$$max\left\{\sum_{bR_ja}[sup(S)]_{jb}, \sum_{bP_ia}[sup(S)]_{ib}\right\} < \epsilon < \sum_{bR_ia}[sup(S)]_{ib}$$

Then there exists $M \in S$ such that $\sum_{bR_ia} M_{ib} > \epsilon$. Since $\sum_{bP_ia} M_{ib} < \epsilon, \sum_{bR_ja} M_{ib} < \epsilon$, we have $M_{ia} > 0$ and $\sum_{bR_ja} M_{ib} < \sum_{bR_ia} M_{ib}$, contradicting to the ordinal fairness of M.

Now suppose that $sup(S)$ is not ex-ante stable, then there exist $i \in N, a \in A$ such that $[sup(S)]_{ib} > 0$ for some $b \in A \cup \{i\}$ with aP_ib, and $[sup(S)]_{ja} > 0$ for some $j \in N \cup \{a\}$ with $i \succ_a j$. Clearly we have $M_{ic} > 0$ for some $c \in A \cup \{i\}$ with aP_ic for all $M \in S$. If $j \in N$, then there exists $M \in S$ with $M_{ja} > 0$, contradicting to the ex-ante stability of M. If $j = a$, then for each $k \in N$, there exists $M^k \in S$ such that

$$\sum_{dP_ka} M^k_{kd} > sup\left\{\sum_{dP_ka} M_{kd} : M \in S\right\} - \frac{1}{N}\left\{1 - \sum_{l\in N}[sup(S)]_{la}\right\}$$

Let $S' = \left\{M^k\right\}_{k\in N}$. Then for each $k \in N$,

$$
\begin{aligned}
[sup(S')]_{ka} =& sup\left\{\sum_{dR_ka} M_{kd} : M \in S'\right\} - sup\left\{\sum_{dP_ia} M_{kd} : M \in S'\right\} \\
<& sup\left\{\sum_{dR_ka} M_{kd} : M \in S\right\} - sup\left\{\sum_{dP_ka} M_{kd} : M \in S\right\} + \frac{1}{N}\left\{1 - \sum_{l\in N}[sup(S)]_{la}\right\} \\
=& [sup(S)]_{ka} + \frac{1}{|N|}\left\{1 - \sum_{l\in N}[sup(S)]_{la}]\right\}
\end{aligned}
$$

Summing over N, we have

$$
\sum_{k\in N}[sup(S')]_{ka} < 1
$$

this implies that $sup(S')$ is wasteful, contradicting to Proposition 3.

Finally, we show that the lattice $(\mathcal{E}, R_N^{sd})$ is distributive. Let $M^x, M^y, M^z \in \mathcal{E}$, $i \in N, a \in A$. For simplicity, denote $k(R) = \sum_{bR_ia} M^k_{ib}$, $k(P) = \sum_{bP_ia} M^k_{ib}$, $k \in \{x, y, z\}$. Then we have

$$
[M^x \wedge_N (M^y \vee_N M^z)]_{ia} = min\left\{x(R), max\left\{y(R), z(R)\right\}\right\} - min\left\{x(P), max\left\{y(P), z(P)\right\}\right\}
$$

$$
[(M^x \wedge_N M^y) \vee_N (M^x \wedge_N M^z)]_{ia} =
$$

$$
max\left\{min\left\{x(R), y(R)\right\}, min\left\{x(R), z(R)\right\}\right\} - max\left\{min\left\{x(P), y(P)\right\}, min\left\{x(P), z(P)\right\}\right\}
$$

Then $[M^x \wedge_N (M^y \vee_N M^z)]_{ia} = [(M^x \wedge_N M^y) \vee_N (M^x \wedge_N M^z)]_{ia}$ by the fact that for real numbers the minimum relation is distributive over the maximum relation, i.e., for any real numbers x, y and z, $min\left\{x, max\left\{y, z\right\}\right\} = max\left\{min\left\{x, y\right\}, min\left\{x, z\right\}\right\}$. Similarly, it can be shown that $M^x \vee_N (M^y \wedge_N M^z) = (M^x \vee_N M^y) \wedge_N (M^x \vee_N M^z)$, using the fact that the maximum relation is distributive over the minimum relation. $\square$

Appendix D: Proofs of Proposition 4 and 5

Proof of Proposition 4 Given any $e = (N, A, R, \succeq)$, let $lim_{k\to\infty} M^k(e) = M^*$. For simplicity we will omit the dependance on e for each $M^k(e)$. We first show M^* is a well-defined random allocation. For any $i \in N$ and $a \in A$, $M^*_{ia} \in [0,1]$, since $M^k_{ia} \in [0,1]$ for each k. For each i, $\sum_{a\in A} M^*_{ia} \leq 1$ since $\sum_{a\in A} M^k_{ia} \leq 1$ for each k. Suppose that for some a, $\sum_{i\in N} M^*_{ia} > 1$. Let $N' = \left\{i \in N : M^*_{ia} > 0\right\}$ and pick ϵ such that

$$0 < \epsilon < min\left\{\frac{1}{|N'|}(\sum_{i\in N} M^*_{ia} - 1), M^*_{ia}\right\}, \forall i \in N'$$

Since $\left\{\sum_{bP_ia} M^k_{ib}\right\}$ is a decreasing sequence for each i, there exists K such that for each $i \in N'$ and $k \geq K$, $\sum_{bP_ia} M^k_{ib} < \sum_{bP_ia} M^*_{ib} + \epsilon$. $\left\{\sum_{bR_ia} M^k_{ib}\right\}$ is also decreasing, so for each $i \in N'$, $\sum_{bR_ia} M^k_{ib} \geq \sum_{bR_ia} M^*_{ib}$. Hence, from the construction of $\left\{M^k\right\}$, it is clear that for each $i \in N'$ and $k \geq K$, any fraction in $[\sum_{bP_ia} M^*_{ib} + \epsilon, \sum_{bR_ia} M^*_{ib}]$ of agent i is either applying to a at step k or is tentatively accepted by a at a step before k. However,

$$\begin{aligned}
\sum_{i\in N'} |[\sum_{bP_ia} M^*_{ib} + \epsilon, \sum_{bR_ia} M^*_{ib}]| &= \sum_{i\in N'} (\sum_{bR_ia} M^*_{ib} - \sum_{bP_ia} M^*_{ib} - \epsilon) \\
&= \sum_{i\in N} M^*_{ia} - |N'|\epsilon \\
&> 1
\end{aligned}$$

A contradiction is reached since a will reject some fraction in $[\sum_{bP_ia} M^*_{ib} + \epsilon, \sum_{bR_ia} M^*_{ib}]$ of some agent $i \in N'$ at step K. Hence M^* is a well-defined random allocation.

Now we show M^* is ex-ante fair. Suppose M^* is not ex-ante stable, then there exist $i \in N, a \in A$ such that $M^*_{ib} > 0$ for some $b \in A \cup \{i\}$ with aP_ib and $M^*_{ja} > 0$ for some $j \in N \cup \{a\}$ with $i \succ_a j$. Obviously, there is a fraction of i that is rejected by a at some step K. Then for all $k > K$, $\sum_{l\in N} M^k_{la} \geq 1$, thus $\sum_{l\in N} M^*_{la} = 1$. This implies that we cannot have $j = a$. If $j \in N$, since $M^*_{ja} > 0$, pick some ϵ such that $0 < \epsilon < M^*_{ja}$. Then there exists K' such that for all $k \geq K'$, $\sum_{bR_ja} M^*_{jb} \leq \sum_{bR_ja} M^k_{jb}$ and $\sum_{bP_ja} M^k_{jb} < \sum_{bP_ja} M^*_{jb} + \epsilon$. Hence for each $k > max\{K, K'\}$, the fractions in $[\sum_{bP_ja} M^*_{jb} + \epsilon, \sum_{bR_ja} M^*_{jb}]$ of j are either applying to a or tentatively accepted by a before step k, which is impossible since $i \succ_a j$ and i has a fraction rejected by a at step K.

Suppose M^* is not ordinally fair, then there exist $i, j \in N, a \in A$ with $i \sim_a j, M^*_{ia} > 0$ and $\sum_{bR_ia} M^*_{jb} < \sum_{bR_ia} M^*_{ib}$. Similar to the previous arguments, we can find some x, y and K such that $\sum_{bR_ia} M^*_{jb} < x < y < \sum_{bR_ia} M^*_{ib}$ and for each $k \geq K$, fractions in $[y, \sum_{bR_ia} M^*_{ib}]$ of i are either applying to a or tentatively accepted by a at a step before k. This is impossible since the fraction $[\sum_{bR_ia} M^*_{jb}, x]$ of j is rejected by a at some step and $[\sum_{bR_ia} M^*_{jb}, x] < [y, \sum_{bR_ia} M^*_{ib}]$.

Finally, we show that M^* is agent-optimal ex-ante fair. For any ex-ante fair allocation M, let μ^M be a F-stable interval matching that induces M. For simplicity, if $\mu^M(i(F)) \subseteq \{(\lambda, a(G)) : a \in A\}$, denote $\mu^M(i(F)) = a$. Then we show the following claim.

Claim 3. If some fraction F of i is rejected by a at some step, then for any $G \subseteq F$ and ex-ante fair allocation M, $\mu^M(i(G)) \neq a$.

Proof of Claim 3. Suppose Claim 3 is not true and let K be the first step that it is not true. Then some fraction F of i is rejected by a at step K, and for some $G \subseteq F$ and ex-ante fair allocation M, $\mu^M(i(G)) = a$. If a fraction F' of i is tentatively accepted by a at step K, then $\mu^M(i(F')) = a$, since K is the first step that Claim 3 is not true and μ^M satisfies $A.6$ (monotonicity). This implies that there exists some fraction F'' of some agent $j \neq i$ that is tentatively accepted by a at step K and $\mu^M(j(F'')) = b \neq a$. Again, by the fact that K is the first step that Claim 3 is not true, we have aP_jb. Since F'' of j is tentatively accepted by a and G of i is rejected at the same step, $j(F'') \succ_{a([0,1])} i(G)$, contradicting to the F-stability of μ^M. $\square$

It can be easily seen that Claim 3 further implies that if some fraction F of i is rejected by a at some step, then for any $G \subseteq F$ and ex-ante fair allocation M, $\mu^M(i(G)) = b$ implies aP_ib. Consider any M^k and ex-ante fair allocation M, suppose for some $i \in N$ and $a \in A$, $\sum_{bR_ia} M^k_{ib} < \sum_{bR_ia} M_{ib}$, then some fraction $F \subseteq [\sum_{bR_ia} M^k_{ib}, \sum_{bR_ia} M_{ib}]$ of i is rejected by a at some step and $\mu^M(i(F)) = b$ for some bR_ia, contradiction. Hence for any M^k and ex-ante fair allocation M, $M^k_i R^{sd}_i M_i$ for all $i \in N$. It follows that $M^* R^{sd}_N M$, so M^* is agent-optimal ex-ante fair. $\square$

Proof of Proposition 5 Given $e = (N, A, R, \succeq) \in E^{HA}$, we show that $f^{SD}(|N!|^{|A|}, e)$ and $f^{EPS}(e)$ are essentially equivalent. Let $C = |N!|^{|A|}$. Suppose the extended PS algorithm terminates at step $\bar{k}$. Let $\{\lambda_k\}_{k=1}^{\bar{k}}$, $\{N_k\}_{k=1}^{\bar{k}}$ and $\{A_k\}_{k=1}^{\bar{k}}$ be the sequences defined in the extended PS. Then $\sum_{k=1}^{\bar{k}} \lambda_k = 1$ and by the construction it can be easily seen that $\lambda_k C$ is a natural number for each $k \in \{1, 2, ..., \bar{k}\}$. Given $k \in \{1, 2, ..., \bar{k}\}$, define

$$I_l(k) = \left\{ i \in N_l : i \notin \cup_{m=l+1}^{k} N_m \text{ } if \text{ } l < k \right\}$$

for each $l \in \{0, 1, ..., k\}$, with $N_0 = N$. Then we have for each $k \geq 2$,

$$\lambda_k = \frac{|B_{N_k}(A_{k-1})| - \sum_{l=0}^{k-2} \left\{ |I_l(k-1) \cap N_k| \sum_{m=l+1}^{k-1} \lambda_m \right\}}{|N_k|}$$

Consider any $\sigma \in \sigma(e^C)$ and any deterministic allocation M^C selected by the serial dictatorship. Let M be the random allocation for e that is generated by M^C. For simplicity, we denote $\gamma(i_t) \in A' \subseteq A$ if $M^C_{i_t a_{t'}} = 1$ implies $a \in A'$. The following statement is the key to the proof.

Claim 4. For any $k \in \{1, 2, ..., \bar{k}\}$ and $l \in \{0, 1, ..., k-1\}$, $i \in I_l(k-1)$ implies $\gamma(i_t) \in B_i(A_{k-1})$ for all $1 + \sum_{m=1}^{l} \lambda_m C \leq t \leq \sum_{m=1}^{k} \lambda_m C$.

Before proving Claim 4, we first show that it implies that $f^{SD}(|N!|^{|A|}, e)$ and $f^{EPS}(e)$ are essentially equivalent. For any k, if $i \in N_k \cap I_l(k-1)$, then in the random allocation M we have $\sum_{a \in B_i(A_{k-1})} M_{ia} \geq \frac{1}{C}(\sum_{m=l+1}^{k} \lambda_m C) = \lambda_k + s_{k-1}(i)$. By the construction of λ_k, it must be the case that $\sum_{a \in B_i(A_{k-1})} M_{ia} = \lambda_k + s_{k-1}(i)$.

To show Claim 4, first consider $k = 1$. By the construction of serial dictatorships, it is sufficient show that there exists some allocation in which each $i_t, i \in N, 1 \leq t \leq \lambda_1 C$, is assigned some a_t with $a \in B_i(A)$. If there does not exist such an allocation, then by Hall's theorem, there exists some $N' \in N, T_i \in \{1, 2, ..., \lambda_1 C\}$ for each $i \in N'$ such that $|\sum_{i \in N'} T_i| > C|B_{N'}(A)|$. It follows that $\lambda_1 C|N'| > C|B_{N'}(A)|$, so $\frac{|B_{N'}(A)|}{|N'|} < \lambda_1$, contradiction.

Now suppose Claim 4 holds for $k-1$, $k \geq 2$. Clearly all the parts of $A \setminus A_{k-1}$ are assigned to parts of $\cup_{m=1}^{k-1} N_m$ by γ. So if $i \in I_l(k-1)$ and $l < k-1$, then $\gamma(i_t) \in B_i(A_{k-1})$ for all $1 + \sum_{m=1}^{l} \lambda_m C \leq t \leq \sum_{m=1}^{k-1} \lambda_m C$. Then by the construction of serial dictatorships, to show Claim 4 for k, it is sufficient to show that there exists some allocation in which each i_t, with $1 + \sum_{m=1}^{l} \lambda_m C \leq t \leq \sum_{m=1}^{k} \lambda_m C$ if $i \in I_l(k-1)$, is assigned some a_t with $a \in B_i(A_{k-1})$ and $t \in \{1, 2, ..., C\}$. Suppose that there does not exist such an allocation. Then by Hall's theorem there exists some $N' \subseteq N, 1 \leq T_i \leq \sum_{m=l+1}^{k} \lambda_m C$ for $i \in I_l(k-1) \cap N'$ such that $\sum_{i \in N'} T_i > C|B_{N'}(A_{k-1})|$. It follows that

$$\sum_{l=0}^{k-1} \left\{ |I_l(k-1) \cap N'| \sum_{m=l+1}^{k} \lambda_m C \right\} > C|B_{N'}(A_{k-1})|$$

Then
$$\sum_{l=0}^{k-2}\left\{|I_l(k-1)\cap N'|\sum_{m=l+1}^{k-1}\lambda_m C\right\}+|N'|\lambda_k C > C|B_{N'}(A_{k-1})|$$

This implies that

$$\frac{|B_{N'}(A_{k-1})|-\sum_{l=0}^{k-2}\left\{|I_l(k-1)\cap N'|\sum_{m=l+1}^{k-1}\lambda_m\right\}}{|N'|}<\lambda_k$$

contradicting to the definition of λ_k. $\square$

Appendix E: Individually rational PS and proof of Proposition 6

Given $e \in E^{HET}$, the **individually rational PS** is defined as follows.

Let $\lambda_0 = 0, r_1(a) = 1$ for each $a \in A$, and $v_1(i) = B_i(A)$ for each $i \in N$. At step k, $k \geq 1$, for each $a \in A$ with $r_k(a) > 0$, define

$$\lambda_k(a) = \frac{r_k(a)}{|\{i \in N : v_k(i) = a\}|}$$

where $\lambda_k(a) = +\infty$ if $\{i \in N : v_k(i) = a\} = \phi$.
For each $N' \subseteq N(e), N' \neq \phi$, let $\bar{N}' = \{i \in N : v_k(i) \in U_{N'}\}$. Define

$$\lambda_k(N') = \frac{\sum_{a\in U_{N'}} r_k(a) - |N'|(1-\sum_{l=0}^{k-1}\lambda_l)}{|\bar{N}'|-|N'|}$$

where $\lambda_k(N') = +\infty$ if $N' = \bar{N}'$. Then

$$\lambda_k = min\left\{min_{a\in A}\left\{\lambda_k(a)\right\}, min_{N'\subseteq N(e)}\left\{\lambda_k(N')\right\}\right\}$$

Each $i \in N$ is assigned λ_k of $v_k(i)$ at step k. For each $a \in A$, $r_{k+1}(a) = r_k(a) - \lambda_k|\{i \in N : v_k(i) = a\}|$. For each $i \in N$,

$$v_{k+1}(i) =$$

$$B_i(\{a \in A : v_k(i)R_i a, r_{k+1}(a) > 0, a \in U_{N'} \textit{ for some } N' \textit{ with } \lambda_k(N') = \lambda_k \textit{ implies } i \in N'\})$$

The process terminates at some step k' if $r_{k'+1}(a) = 0$ for all $a \in A$.

Proof of Proposition 6 Given $e = (N, A, R, \succeq) \in E^{HET}$, we show that $f^{IRSD}(((|N|!)^{|A|^2}, e)$ is single-valued and $f^{IRSD}(((|N|!)^{|A|^2}, e) = f^{IRPS}(e)$. Let $C = (|N|!)^{|A|^2}$. Suppose the individually rational PS algorithm terminates at step $\bar{k}$. Let $\{\lambda_k\}_{k=0}^{\bar{k}}$, $\{r_k(a)\}_{k=1}^{\bar{k}}, a \in A$, and $\{v_k(i)\}_{k=1}^{\bar{k}}, i \in N$ be the sequences defined in the individually rational PS. Then $\sum_{k=1}^{\bar{k}} \lambda_k = 1$ and by the construction it can be easily seen that $\lambda_k C$ is a natural number for each $k \in \{1, 2, ..., \bar{k}\}$.

Consider any $\sigma \in \sigma(e^C)$ and the deterministic allocation M^C selected by the individually rational serial dictatorship. Let M be the random allocation for e that is generated by M^C. For simplicity, we denote $\gamma(i_t) = a$ if $M^C_{i_t a_{t'}} = 1$ for some t'. Then it is sufficient to show the following claim.

Claim 5. For any $k \in \{1, 2, ..., \bar{k}\}$ and $i \in N$, $\gamma(i_t) = v_k(i)$ for all t with $\sum_{m=0}^{k-1} \lambda_m C + 1 \leq t \leq \sum_{m=0}^{k} \lambda_m C$.

We first show that Claim 5 holds for $k = 1$. By the construction of the individually rational serial dictatorship, it is sufficient to show that there exists an allocation in which i_t is assigned some part of $v_1(i)$ for all i and t with $1 \leq t \leq \lambda_1 C$, and for all $i \in N(e)$, $t > \lambda_1 C$, each i_t is assigned some part of an object in U_i. Suppose there does not exist such an allocation. Then by Hall's theorem, there exist some $N^1 \subseteq N$, $(T_i^1)_{i \in N^1}$, $T_i^1 \in \{1, ..., \lambda_1 C\}$ for each $i \in N^1$, and $N^2 \subseteq N(e)$, $(T_i^2)_{i \in N^2}$, $T_i^2 \in \left\{1, ..., \sum_{m=2}^{\bar{k}} \lambda_m C\right\}$ for each $i \in N^2$, such that

$$\sum_{i \in N^1} T_i^1 + \sum_{i \in N^2} T_i^2 > |\left\{\cup_{i \in N^1} \{v_1(i)\}\right\} \cup U_{N^2}| C$$

Then

$$|N^1| \lambda_1 C + |N^2| \sum_{m=2}^{\bar{k}} \lambda_m C > |\left\{\cup_{i \in N^1} \{v_1(i)\}\right\} \cup U_{N^2}| C$$

Clearly $N^1 \neq \phi$. If $N^2 = \phi$, then we have $\frac{|\cup_{i \in N^1} \{v_1(i)\}|}{|N^1|} < \lambda_1$, contradicting to the definition of λ_1. If $N^2 \neq \phi$, let $N^3 = \left\{i \in N^1 : v_1(i) \in U_{N^2}\right\}$ and $N^4 = N^1 \setminus N^3$. We have

$$|N^3| \lambda_1 + |N^4| \lambda_1 + |N^2| \sum_{m=2}^{\bar{k}} \lambda_m > |\cup_{i \in N^4} \{v_1(i)\}| + |U_{N^2}|$$

Then,

$$|N^2 \cup N^3| \lambda_1 + |N^4| \lambda_1 + |N^2| - |N^2| \lambda_1 > |\cup_{i \in N^4} \{v_1(i)\}| + |U_{N^2}|$$

If $|N^4|\lambda_1 > |\cup_{i\in N^4}\{v_1(i)\}|$, then $\frac{|\cup_{i\in N^4}\{v_1(i)\}|}{|N^4|} < \lambda_1$, contradicting to the definition of λ_1. If $|N^4|\lambda_1 \leq |\cup_{i\in N^4}\{v_1(i)\}|$, then

$$|N^2 \cup N^3|\lambda_1 + |N^2| - |N^2|\lambda_1 > |U_{N^2}|$$

and $N^3 \neq \phi$. Hence

$$\frac{|U_{N^2}| - |N^2|}{|N^2 \cup N^3| - |N^2|} < \lambda_1$$

contradiction.

Suppose Claim 5 holds for some $k \geq 1$. Establishing Claim 5 for $k+1$ completes the proof. By the construction of the individually rational serial dictatorship, it is sufficient to show that there exists some allocation in which i_t is assigned some part of $v_{k+1}(i)$ for all i and t with $\sum_{m=1}^{k}\lambda_m C + 1 \leq t \leq \sum_{m=1}^{k+1}\lambda_m C$, and for each $i \in N(e), t > \sum_{m=1}^{k+1}\lambda_m C$, i_t is assigned some part of an object in U_i. Suppose there does not exist such an allocation. Then by Hall's theorem and the fact that Claim 5 holds for k, there exist $N^1 \subseteq N, 1 \leq T_i^1 \leq \lambda_k C$ for each $i \in N^1$, and $N^2 \subseteq N(e)$, $1 \leq T_i^2 \leq \sum_{m=k+2}^{\bar{k}}\lambda_m C$ for each $i \in N^2$, such that

$$\sum_{i\in N^1} T_i^1 + \sum_{i\in N^2} T_i^2 > \sum_{a\in\left\{\cup_{i\in N^1}\{v_{k+1}(i)\}\right\}\cup U_{N^2}} r_{k+1}(a)C$$

Then

$$|N^1|\lambda_{k+1}C + |N^2|\sum_{m=k+2}^{\bar{k}}\lambda_m C > \sum_{a\in\left\{\cup_{i\in N^1}\{v_{k+1}(i)\}\right\}\cup U_{N^2}} r_{k+1}(a)C$$

Clearly $N^1 \neq \phi$. If $N^2 = \phi$, then we have

$$\frac{\sum_{a\in\cup_{i\in N^1}\{v_{k+1}(i)\}} r_{k+1}(a)}{|N^1|} < \lambda_{k+1}$$

contradicting to the definition of λ_{k+1}. Suppose $N^2 \neq \phi$. Let $N^3 = \left\{i \in N^1 : r_{k+1}(i) \in U_{N^2}\right\}$ and $N^4 = N^1 \setminus N^3$. Then

$$|N^3|\lambda_{k+1} + |N^4|\lambda_{k+1} + |N^2|\sum_{m=k+2}^{\bar{k}}\lambda_m > \sum_{a\in U_{N^2}} r_{k+1}(a) + \sum_{a\in\cup_{i\in N^4}\{v_{k+1}(i)\}} r_{k+1}(a)$$

If $|N^4|\lambda_{k+1} > \sum_{a\in\cup_{i\in N^4}\{v_{k+1}(i)\}} r_{k+1}(a)$, then clearly this contradicts to the definition of λ_{k+1}. If $|N^4|\lambda_{k+1} \leq \sum_{a \in \cup_{i \in N^4}\{v_{k+1}(i)\}} r_{k+1}(a)$, then

$$|N^3|\lambda_{k+1} + |N^2| \sum_{m=k+2}^{\bar{k}} \lambda_m > \sum_{a \in U_{N^2}} r_{k+1}(a)$$

Hence

$$|N^2 \cup N^3|\lambda_{k+1} + |N^2| \sum_{m=k+1}^{\bar{k}} \lambda_m - |N^2|\lambda_{k+1} > \sum_{a \in U_{N^2}} r_{k+1}(a)$$

This implies $N^3 \neq \phi$ and

$$\frac{\sum_{a \in U_{N^2}} r_{k+1}(a) - |N^2| \sum_{m=k+1}^{\bar{k}} \lambda_m}{|N^2 \cup N^3| - |N^2|} < \lambda_{k+1}$$

contradiction. $\square$

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