A Theory of Fair Random Allocation Under Priorities*

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Abstract

In the allocation of indivisible objects under weak priorities, a common practice is to break the ties using a lottery and randomize over deterministic mechanisms. Such randomizations usually lead to unfairness and inefficiency ex-ante. We propose and study the concept of ex-ante fairness for random allocations, extending some key results in the one-sided and two-sided matching markets. It is shown that the set of ex-ante fair random allocations forms a complete lattice under first-order stochastic dominance relations, and the agent-optimal ex-ante fair mechanism includes both the deferred acceptance algorithm and the probabilistic serial mechanism as special cases. Instead of randomizing over deterministic mechanisms, our mechanism is constructed using the division method, a new general way of constructing random mechanisms from deterministic mechanisms. As additional applications, we demonstrate that several previous extensions of the probabilistic serial mechanism can be constructed from existing deterministic mechanisms.

Keywords: indivisible object, weak priority, random allocation, fairness, deferred acceptance algorithm, probabilistic serial mechanism

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1 Introduction

We study "priority-augmented" allocation of indivisible objects without monetary transfers: while agents have ordinal preferences over some heterogeneous objects, each object also has its own priority ranking over the agents. *School choice* (Abdulkadiroğlu and Sönmez, 2003b) is one of the most popular applications of this problem in practice. The nature of priority-augmented allocation is closely related to the classical two-sided matching (Gale and Shapley, 1962), and in our context stability is regarded as a key fairness consideration, which requires the differences in priorities be respected, i.e., no agent envies another's assignment for which the first agent has a higher priority. In the case of strict preferences and strict priorities, the *deferred acceptance algorithm* (DA) from Gale and Shapley (1962) is often considered as the best mechanism, since it is agent-optimal stable and strategy-proof. In this study, we consider the more general case with weak priorities, where the appropriate axioms to be imposed and the optimal choice of mechanism are much less obvious. Ties in priorities are indeed commonly observed in practice.[1] Moreover, this framework includes the classical *house allocation* problem (Hylland and Zeckhauser, 1979) as a special case, by treating all the agents as equally ranked at every object.

In the presence of ties in priorities, the additional fairness consideration regarding agents with equal priority requires the use of random allocations. The most common way of generating a random mechanism, both in theory and in practice, is to randomize over deterministic mechanisms. In particular, we can first break the ties in priorities using a randomly selected ordering of agents, then apply a deterministic mechanism. Some familiar examples include random serial dictatorship (RSD) in house allocation (Abdulkadiroğlu and Sönmez, 1998), top trading cycles mechanism with random ordering in house allocation with existing tenants (Abdulkadiroğlu and Sönmez, 1999, Sönmez and Ünver, 2005), and DA with random tie-breaking in school choice (Abdulkadiroğlu and Sönmez, 2003b). It is well-known that random mechanisms constructed in this way can preserve the strategy-proofness of the deterministic mechanisms, but the outcomes usually suffer from inefficiency and unfairness from the ex-ante perspective. In light of this issue, two notable studies design new random mechanisms that assign probability shares of objects to agents directly, based on ordinal preferences. In house

[1] For instance, in a school choice program, students are often prioritized based on only a few criteria (e.g. the walk zone and sibling criteria), and hence many students may have the same priority at a school.

allocation, Bogomolnaia and Moulin (2001) introduce the *probabilistic serial mechanism* (PS) that achieves (first-order) stochastic-dominance efficiency and stochastic-dominance envy-freeness.[2] In the general case with weak priorities, Kesten and Ünver (2015) propose a fairness concept from the ex-ante perspective, *strong ex-ante stability*, and construct the *fractional deferred acceptance mechanism*, which is agent-optimal strongly ex-ante stable.[3]

In this study, we first take a similar approach as Kesten and Ünver (2015), and propose a new and normatively appealing fairness concept, *ex-ante fairness*, that is generally not satisfied by existing random mechanisms (Section 3). Ex-ante fairness is defined as the combination of two separate axioms: *ex-ante stability* and *ordinal fairness*. Ex-ante stability is studied in Roth et al. (1993) in the context of two-sided matching, and introduced to priority-augmented allocation by Kesten and Ünver (2015). It requires that if agent i is ranked higher than agent j by object a, then j cannot receive a with a positive probability, unless i receives an object weakly better than a with probability one. Hence, it is a direct extension of the stability concept to the probabilistic setting. Ordinal fairness, on the other hand, requires that if agents i and j are ranked equally by object a, and i has a positive probability of receiving it, then her probability of receiving an object weakly better than a cannot be larger than j's probability of receiving an object weakly better than a. In other words, the allocation of an object among equally ranked agents should try to equalize each one's probability of obtaining a weakly better object. This is an adaption of the original ordinal fairness axiom defined by Hashimoto et al. (2014) in house allocation to our context,[4] and conforms to the idea of *compensation*, a general principle of fairness and social justice (Moulin, 2004).

We then introduce the *deferred consumption mechanism* (DC) to show that an ex-ante fair random allocation always exists, which naturally combines features of DA and PS (Section 4). This mechanism is defined through the following iterative procedure: in each step, every agent proposes to consume objects during certain time intervals (starting from proposing to consume her favorite object during the whole time interval $[0, 1]$ in the first step), and each object tentatively accepts some proposals based on its

[2] PS determines a random allocation through an "eating" or consumption process, where the agents simultaneously consume the probability shares of their best available objects at the unit rate during the unit time interval.

[3] These two mechanisms are not related: the fractional deferred acceptance mechanism is not reduced to PS in the special case of house allocation.

[4] They show that ordinal fairness, together with individual rationality and non-wastefulness, characterizes PS.

priority ranking as well as the principle of first-come first-serve;[5] if an agent's proposal to consume some object during time interval $[x, y]$ is rejected, she proposes to consume her next best option during $[x, y]$ in the next step. This procedure may not be finite, but we define a tentative assignment for each step and show the sequence of tentative assignments converges to the *agent-optimal ex-ante fair random allocation* (Theorem 1), i.e., the unique ex-ante fair random allocation that stochastically dominates every other ex-ante fair random allocation for all agents. DC is reduced to DA in the special case of strict priorities, and to PS in the special case of house allocation.

We further explore the properties of ex-ante fair random allocations and show that they are well-behaved (Theorem 2 in Section 5). The set of ex-ante fair random allocations is a complete lattice under the stochastic dominance relations of the agents, which extends Conway's lattice theorem of stable matchings (Knuth, 1976). We also establish a generalized version of the rural hospital theorem (McVitie and Wilson, 1970, Roth, 1984, 1986): each agent or object's probability of being assigned is constant across all the ex-ante fair random allocations. Moreover, if this probability is less than one, then this agent or object receives the same lottery at every ex-ante fair random allocation. Finally, in some cases, objects are not entirely passive and they may have intrinsic preferences that are aligned with their priorities.[6] In this context, we show that agents and objects have opposite interests over ex-ante fair random allocations, and each ex-ante fair random allocation is stochastic-dominance efficient for the two sides of the market. These results generalize the classical properties of stable matchings in the two-sided matching theory, and they do not hold for stable deterministic allocations when priorities are weak.

Next, to better understand our main results above, we introduce a new general method of constructing random mechanisms from deterministic mechanisms (Section 6). This method is referred to as *division*, and the idea is as follows. Given some positive integer q, we divide (the claim of) each agent i into q parts, $i^1, \dots, i^q$, and each object a into q parts as well. Then the *(finitely) divided problem* consists of these agent parts and object parts. For each part of an object a, we specify a priority ranking over agent parts

[5] For example, in some step, Alice, Bob and Carl propose to consume an object a during time intervals $[0, 1]$, $[\frac{1}{2}, 1]$ and $[0, 1]$ respectively. Alice and Bob are equally ranked by a, and they are ranked higher than Carl. Then Carl's proposal is rejected first. If Alice and Bob consume a starting from the time at which they come ($t = 0$ for Alice and $t = \frac{1}{2}$ for Bob), then a will be exhausted at $t = \frac{3}{4}$. Hence, a tentatively accepts Alice's proposal $[0, \frac{3}{4}]$ and Bob's proposal $[\frac{1}{2}, \frac{3}{4}]$, and rejects the rest of their proposals.

[6] For instance, in school choice, a school's priorities may be determined based on students' academic attainments, and hence largely reflect its preferences over students.

such that i^x is ranked weakly higher than j^y if (1) i is ranked higher than j by a, or, (2) i and j are equally ranked by a and $x \leq y$. We also specify preferences of each agent part over the object parts, according to the preferences in the original problem. Then, the divided problem is itself a priority-augmented allocation problem. A deterministic allocation for the divided problem generates a random allocation for the original problem, where an agent i's probability of being assigned an object a is the proportion of i's parts that are assigned a's parts. We give some immediate and preliminary results, and argue that random allocations generated in this way can have better efficiency and fairness properties from the ex-ante perspective, compared to the usual randomization method.[7]

The finite division framework just described will be formally used in our other results later. Now, to give an alternative perspective on ex-ante fairness, we need to envision a *continuum divided problem*, where each agent and object is divided into a continuum of parts with measure one. Then, every ex-ante fair random allocation is generated by a stable deterministic allocation in the continuum divided problem. This connection with stability helps explain some of the key results in Theorem 2. Moreover, DC is essentially generated by applying DA to continuum divided problems. Note that DC is not strategy-proof, as PS is not strategy-proof. More generally, one drawback of the division method is that it does not preserve the incentive compatibility of a deterministic mechanism, which can also be seen in all the random mechanisms discussed below.

In the end, we present some other applications of the division method (Section 7). While DC provides a generalization of PS to priority-augmented allocation using this method, we show that several other previous generalizations of PS can also be obtained by applying well-known deterministic mechanisms to (finitely and appropriately) divided problems.[8] First, in house allocation under weak preferences, the *extended PS* solution by Katta and Sethuraman (2006) can be obtained from the *serial dictatorships* defined for weak preferences by Svensson (1994) (Proposition 1). In this context, we also show the division method can potentially generate new stochastic-dominance efficient and stochastic-dominance envy-free mechanisms. Second, in allocation problems with private endowments, the generalized PS by Yılmaz (2010) can be obtained from

[7]Although the interpretation is different, the idea of the division method in the special case of house allocation first appears in Kesten (2009). See the discussion regarding Proposition 1 in Section 7.1.

[8]Finite division is sufficient to reconstruct these generalizations of PS, but the number q has to be chosen correctly when we divide every agent and object into q parts. For instance, to allocate one object fairly between two agents with equal priority, it is obvious that q must be an even number. The choice of q in the following results generally follows a principle that extends this very simple intuition.

a variation of serial dictatorships that preserves stability,[9] and the generalized PS by Zhang (2017) can be obtained from the *top trading cycles mechanism* (Abdulkadiroğlu and Sönmez, 2003b) (Proposition 2). Then, for house allocation with multi-unit demands, the two generalizations of PS by Kojima (2009) and Heo (2014) can be obtained from two different forms of serial dictatorships under multi-unit demands (Pápai, 2000, Bogolmonaia et al., 2014), respectively. Finally, in the general case with weak priorities, motivated by the inefficiency of the randomized *Boston mechanism* (Harless, 2018), we construct a probabilistic version of this mechanism using the division method, which restores some key desirable features of the deterministic Boston mechanism under strict priorities.

In the above results, we demonstrate that many interesting random mechanisms have their foundations in deterministic mechanisms. Therefore, our main contribution to the random assignment literature is not only an extension of PS to handle priorities, but also an attempt to broadly interpret the random mechanisms related to PS as applications of the division method, which has the potential to generate new mechanisms as well.

1.1 Related Literature

As can be seen from the previous discussions, this paper is related to several strands of the matching and market design literature, including both the (deterministic and random) one-sided matching studies, and the (deterministic and random) two-sided matching studies. In this subsection we discuss related results that are not mentioned above.

First, after the seminal work of Gale and Shapley (1962), the key insights in the two-sided matching problem have been generalized in various directions. In particular, both Roth et al. (1993) and Alkan and Gale (2003) establish the lattice structure of stable random matchings. The former assumes strict orderings from both sides of the market, and the latter takes a revealed preference approach. The properties of ex-ante fair random allocations are logically independent of their results.

The introduction of PS by Bogomolnaia and Moulin (2001) leads to a growing lit-

[9] Yılmaz (2009) considers allocation problems with private endowments and weak preferences, and proposes a new solution that extends the mechanisms in Yılmaz (2010) and Katta and Sethuraman (2006). This solution is further extended by Athanassoglou and Sethuraman (2011) to allocation problems with fractional endowments.

erature on random assignment. Several studies contribute to a better understanding of this new and intuitive mechanism. In addition to axiomatic characterizations given by Bogomolnaia and Heo (2012) and Hashimoto et al. (2014), Che and Kojima (2010) establish the asymptotic equivalence of PS and RSD in large markets, and Bogomolnaia (2015) provides a new and welfarist interpretation of (extended) PS. In addition, Kesten (2009) shows that PS can be viewed as a form of RSD or top trading cycles mechanism. His result regarding PS and RSD is in the same vein as our Proposition 1, and embeds the idea of division.

The ex-ante inefficiency and unfairness of the randomization method are first observed by Bogomolnaia and Moulin (2001). They show that RSD is not stochastic-dominance efficient or stochastic-dominance envy-free. Erdil (2014) shows that RSD (or more generally, DA with random tie-breaking) is not even non-wasteful and can be stochastically dominated by another strategy-proof random mechanism. Random tie-breaking also imposes artificial stability constraints, leading to additional efficiency loss. Such issues are studied in Erdil and Ergin (2008) and Abdulkadiroğlu et al. (2009). Furthermore, Kesten and Ünver (2015) show that existing random mechanisms do not satisfy strong ex-ante stability, which is the combination of ex-ante stability and *no ex-ante discrimination*. The latter axiom essentially requires that the allocation of an object among equally ranked agents should try to equalize their probabilities of receiving this object.[10] In contrast, our ordinal fairness concept requires that the allocation should also take into account these agents' chances of receiving better objects, and try to equalize their probabilities of receiving an object weakly better than this one. Our deferred consumption mechanism closely resembles their fractional deferred acceptance mechanism, although there is no logical relation between the two.[11]

Finally, as far as we know, there are two other studies that also extend PS to settings with priorities, and unify PS and DA. Afacan (2018) considers a different and novel model with *random priorities*, where a probability distribution over all strict priority structures is given as a component of the allocation problem. He proposes the *constrained probabilistic serial mechanism* to achieve the desiderata relevant for such problems. This mechanism is reduced to DA when the priorities are deterministic, and

[10] More precisely, it says that if agents i and j are equally ranked by an object a, then i cannot receive less of a than j, unless i's probability of receiving an object weakly better than a is one.

[11] In each step of the fractional deferred acceptance mechanism, fractions of agents propose to objects. Each object tentatively accepts fractions of agents with higher priorities first, and for agents with equal priority, it tries to accept an equal fraction of each one. Thus, the main difference between this procedure and deferred consumption is that in our case we keep track of which fractions of agents propose.

to PS when every two strict priority structures are equally likely. Aziz and Brandl (2021) extend PS and introduce the *vigilant eating rule* that applies to weak preferences and almost arbitrary constraints.[12] Whenever the set of random allocations satisfying the constraints is non-empty and closed, it chooses a constrained stochastic-dominance efficient allocation. Therefore, given a weak priority structure, if some stability concept is imposed as constraints, then the rule is equivalent to DA in the case of strict priorities, and to PS in the case of house allocation.[13]

2 Preliminaries

Let N be a finite set of **agents** and A a finite set of **objects**. Each agent $i \in N$ has a complete and transitive **preference relation** R_i on $A \cup \{i\}$, with P_i and I_i denoting its asymmetric and symmetric components, respectively. We also assume that R_i is anti-symmetric for every $i \in N$, i.e., preferences are strict, unless otherwise specified. A **preference profile** $R = (R_i)_{i \in N}$ is a list of individual preferences. Each object $a \in A$ has a complete and transitive **priority ordering** $\succeq_a$ on N, with $\succ_a$ and $\sim_a$ denoting its asymmetric and symmetric components, respectively. A **priority structure** $\succeq = (\succeq_a)_{a \in A}$ is a profile of priority orderings. Then, a **priority-augmented allocation problem**, or simply a **problem**, is summarized as $p = (N, A, R, \succeq)$.[14]

For a given problem $p = (N, A, R, \succeq)$, a **random allocation**, or simply an **allocation**, is denoted by a $|N| \times |A|$ matrix M such that $M_{ia} \geq 0$, $\sum_{b \in A} M_{ib} \leq 1$, and $\sum_{j \in N} M_{ja} \leq 1$ for all $i \in N$ and $a \in A$, where M_{ia} represents the probability that agent i is assigned object a. For each $i \in N$, let $M_i = (M_{ia})_{a \in A \cup \{i\}}$ denote the lottery obtained by i under the allocation M, where $M_{ii} = 1 - \sum_{a \in A} M_{ia}$ is the probability that i receives her outside option. Similarly, for each $a \in A$, let $M_a = (M_{ia})_{i \in N \cup \{a\}}$, where $M_{aa} = 1 - \sum_{i \in N} M_{ia}$. M is a **deterministic allocation** if $M_{ia} \in \{0, 1\}$ for all $i \in N$ and $a \in A$. For ease of exposition, we also use a one-to-one function $\mu : N \to A \cup N$, where $\mu(i) \in A \cup \{i\}$ for all $i \in N$, to denote a deterministic allocation. By the Birkhoff-von Neumann theorem (Birkhoff,

[12] Budish et al. (2013) and Balbuzanov (2019) also study extensions of PS to allocation with constraints.

[13] See Manjunath (2017) and Aziz and Klaus (2019) for various extensions of the stability concept to the probabilistic setting.

[14] For simplicity, we assume that there is only one copy of each object. That is, we focus on the *one-to-one* setting. Whether the total available probability shares of each object is one or some other positive integer is not important to our analysis of the ex-ante properties of random allocations. Therefore, all the main results in Section 4 and Section 5 can be easily extended to the *many-to-one* setting where there are multiple copies of each object.

1946, von Neumann, 1953), every random allocation can be represented as a lottery over deterministic allocations.[15]

For $i \in N$ and $a \in A \cup \{i\}$, let $F(R_i, a, M) = \sum_{b \in A \cup \{i\}: bR_i a} M_{ib}$ denote the probability that agent i is assigned an option weakly better than a under the allocation M. Then let $F(P_i, a, M) = \sum_{b \in A \cup \{i\}: bP_i a} M_{ib}$. Similarly, for $a \in A$ and $i \in N$, let $F(\succeq_a, i, M) = \sum_{j \in N: j \succeq_a i} M_{ja}$ and $F(\succ_a, i, M) = \sum_{j \in N: j \succ_a i} M_{ja}$. An allocation M is **individually rational** if $F(R_i, i, M) = 1$ for all $i \in N$. It is **non-wasteful** if $M_{aa} > 0$ implies $F(R_i, a, M) = 1$ for all $i \in N$ and $a \in A$. Given any two allocations M and M', each agent i can compare the lotteries M_i and M'_i using the first-order stochastic dominance relation R_i^{sd}: $M_i R_i^{sd} M'_i$ if $F(R_i, a, M) \geq F(R_i, a, M')$ for all $a \in A \cup \{i\}$. Let $M R_N^{sd} M'$ if $M_i R_i^{sd} M'_i$ for all $i \in N$. M **Pareto dominates** M' if $M R_N^{sd} M'$ and we do not have $M' R_N^{sd} M$. Then, an allocation is stochastic dominance efficient, or **sd-efficient**, if it cannot be Pareto dominated by any allocation. We also define the first-order stochastic dominance relations for objects. For each $a \in A$, let $M_a \succeq_a^{sd} M'_a$ if $F(\succeq_a, i, M) \geq F(\succeq_a, i, M')$ for all $i \in N$. In this case, object a has better chances of being assigned to higher ranked agents under M than under M'. Let $M \succeq_A^{sd} M'$ if $M_a \succeq_a^{sd} M'_a$ for all $a \in A$.

A deterministic allocation μ is **efficient** if it can not be Pareto dominated by any other deterministic allocation.[16] It is **stable** if it is individually rational, non-wasteful, and there is *no justified-envy*, i.e., there do not exist $i, j \in N$ such that $\mu(j) P_i \mu(i)$ and $i \succ_{\mu(j)} j$. A stronger fairness notion ensures that a deterministic allocation respects not only the differences but also the indifferences in priorities: μ is **strongly stable** if it is individually rational, non-wasteful, and there do not exist $i, j \in N$ such that $\mu(j) P_i \mu(i)$ and $i \succeq_{\mu(j)} j$. Applying the *deferred acceptance algorithm* (DA) from Gale and Shapley (1962) after ties in priorities are broken in any way yields a stable deterministic allocation. However, a strongly stable deterministic allocation may not exist. This fact also suggests that in general random allocations are needed to restore fairness (regarding ties in priorities).

A **random mechanism**, or simply a **mechanism**, is a function f that assigns an allocation $f(p)$ to each problem p. If $f(p)$ is deterministic for each p, then f is also called a **deterministic mechanism**. f is said to satisfy a certain property defined above if $f(p)$ satisfies this property for all p. Finally, f is **strategy-proof** if, for each agent, truth-telling

[15] There could be multiple lottery representations of a random allocation. In addition, see Budish et al. (2013) for a maximal generalization of this theorem.

[16] If a random allocation is sd-efficient, then it can only be represented as lotteries over efficient deterministic allocations. However, the converse is not true. See Abdulkadiroğlu and Sönmez (2003a) and Kesten et al. (2017) for characterizations of sd-efficiency.

yields a lottery that first-order stochastically dominates the lottery obtained from reporting any other preferences: for any $p = (N, A, R, \succeq)$, $i \in N$ and $p' = \big(N, A, (R'_i, R_{-i}), \succeq\big)$, we have $f_i(p) R_i^{sd} f_i(p')$.

3 Ex-Ante Fairness

We propose a concept of fairness for random allocations from the ex-ante perspective. Consider a problem $p = (N, A, R, \succeq)$. First, we want a random allocation to respect the differences in priorities, such that the assignment of the probability shares of each object always satisfies the demands of higher ranked agents first.

Definition 1. *A random allocation M is* ***ex-ante stable*** *if it is individually rational, non-wasteful, and there do not exist $i, j \in N$ and $a \in A$ such that $i \succ_a j, M_{ja} > 0$, and $F(R_i, a, M) < 1$.*

This notion is discussed in Roth et al. (1993) in the context of two-sided matching with strict preferences on both sides of the market, and is introduced to priority-augmented allocation problems with weak priorities by Kesten and Ünver (2015). It is a direct generalization of the stability concept to random allocations, from the ex-ante perspective. In particular, the last requirement in the definition is a probabilistic version of the no justified-envy condition: if agent i has a higher priority than agent j at object a, then j cannot receive a positive probability share of a, unless i can receive an outcome weakly better than a for sure. A weaker stability notion for random allocations is *ex-post stability*. An allocation is **ex-post stable** if it can be represented as a lottery over stable deterministic allocations. An ex-ante stable allocation is ex-post stable, and it can only be represented as lotteries over stable deterministic allocations (Kesten and Ünver, 2015). On the other hand, an ex-post stable allocation is individually rational, but may not satisfy the other two requirements in the definition of ex-ante stability.

Second, we also want a random allocation to respect the indifferences in priorities. To this end, we need another fairness condition that deals with the ties. One notion in this regard is *equal treatment of equals*, which requires that every two agents with the same preferences and the same priorities at all objects should be assigned the same lottery. While it is a more appropriate restriction in the special case of *house allocation*, where all the agents are ranked equally by each object, it is a weak requirement in general priority-augmented allocation, as two agents can differ not only in their preferences but also in their priorities. For instance, if any two agents have the same priority at every object except one, then, regardless of the preferences, equal treatment of equals is satisfied by every deterministic allocation, and fairness considerations regarding equal priorities are clearly ignored. Therefore, in light of the rich priority domain, we define fairness "*locally*", and impose restrictions on the allocation of each single object among the agents equally ranked by this object.

Definition 2. *A random allocation M is* ***ordinally fair*** *if for any $i, j \in N$ and $a \in A$ with $i \sim_a j$, $M_{ia} > 0$ implies $F(R_j, a, M) \geq F(R_i, a, M)$.*

This concept is first introduced by Hashimoto et al. (2014) for house allocation. They show that, together with individual rationality and non-wastefulness, it characterizes the *probabilistic serial mechanism* (PS) from Bogomolnaia and Moulin (2001). We extend it to the setting with priorities. If $i \sim_a j$, $M_{ia} > 0$ and $F(R_j, a, M) < F(R_i, a, M)$, some probability shares of a can be transferred from i to j to reduce the differences in their probabilities of receiving an object weakly better than a. Therefore, ordinal fairness essentially requires that the allocation of an object among agents in the same priority class should try to equalize their probabilities of receiving a weakly better object (or, equivalently, their chances of receiving a strictly worse outcome). Consequently, an agent with a smaller probability of receiving a strictly better object would generally be assigned a larger share of this object.

Ordinal fairness follows an important general principle of fairness and social justice, *compensation*, which says the allocation of resources should compensate for the differences in those primary individual characteristics, and equalize the shares of the higher-order characteristic (Moulin, 2004).[17] Take the school choice problem as a concrete example, and consider the allocation of the shares of one particular school, a, among some students with equal priority at a. Their different chances of being admitted to a strict better school are attributed to the differences in their preferences, priorities at other schools, the competition levels at other schools, and many other factors. We believe that these differences are relevant for considering the fair allocation of a among them, and the allocation should try to equalize their shares of the higher-order characteristic, i.e., their chances of being admitted to a school at least as good as a.

The two axioms defined above constitute the central concept in this paper.

[17] See Moulin (2004) for detailed and formal discussions regarding four principles of fairness: exogenous rights, compensation, reward, and fitness.

Definition 3. *A random allocation M is* ***ex-ante fair*** *if it is ex-ante stable and ordinally fair.*

Existing random mechanisms generally fail to be ex-ante fair. For example, a widely-used mechanism in school choice is *DA with single tie-breaking* (Abdulkadiroğlu et al., 2009): an ordering of the agents is picked from the uniform distribution to break the ties in priorities before DA is applied. Kesten and Ünver (2015) show that it does not satisfy the probabilistic version of the no justified-envy condition, and is thus not ex-ante stable. Below, we give a simple example in which it is not ordinally fair. We also present all the ex-ante fair allocations in this example.

Example 1. *Suppose that $N = \{1,2,3\}$ and $A = \{a,b,c\}$. The preferences, priorities, and the allocation selected by DA with single tie-breaking are given as follows:*

R_1	R_2	R_3
b	b	a
c	a	c
a	c	b

$\succeq_a$	$\succeq_b$	$\succeq_c$
1	3	2
$2,3$	$1,2$	1
		3

	a	b	c
1	0	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$
3	$\frac{5}{6}$	0	$\frac{1}{6}$

The above allocation, denoted as M, is not ordinally fair: we have $2 \sim_a 3$, $M_{3a} > 0$, and $F(R_2,a,M) = \frac{2}{3} < \frac{5}{6} = F(R_3,a,M)$. In this case, all the probability shares of a are allocated between these two agents, but agent 2 receives too few of a.

There exists a continuum of ex-ante fair allocations for this problem. For every $x \in [0,\frac{1}{2}]$, the allocation $M(x)$ defined below is ex-ante fair.

	a	b	c
1	$1-3x$	x	$2x$
2	x	x	$1-2x$
3	$2x$	$1-2x$	0

if $0 \leq x \leq \frac{1}{3}$

	a	b	c
1	0	x	$1-x$
2	$\frac{1}{2}(1-x)$	x	$\frac{1}{2}(1-x)$
3	$\frac{1}{2}(1+x)$	$1-2x$	$\frac{1}{2}(3x-1)$

if $\frac{1}{3} < x \leq \frac{1}{2}$

Note that $M(x) R_N^{sd} M(y)$ if $x > y$. That is, the ex-ante fair allocation $M(x)$ becomes better for all agents as x increases, with the best one $M(\frac{1}{2})$ and the worst one $M(0)$.

4 Deferred Consumption

In this section, we construct a mechanism to establish the existence of an ex-ante fair allocation. In fact, the mechanism always selects the *best* ex-ante fair allocation for the agents. Formally, given a problem $p = (N, A, R, \succeq)$, we say an allocation M is **agent-optimal ex-ante fair** if it is ex-ante fair, and $MR_N^{sd}M'$ for every ex-ante fair allocation M'.

From previous discussions, it is already known that such mechanism should be reduced to DA in the special case of strict priorities, and to PS in the special case of house allocation.[18] In each step of DA, agents propose to objects, and every object tentatively accepts a proposer based on priorities. The actual acceptance is *deferred* to the last step. In PS, the agents *consume* (or "eat") the objects simultaneously at the unit rate during the unit time interval. Our mechanism is defined by a procedure of *deferred consumption* that combines the above features of DA and PS. In each step, the agents propose to consume objects during certain time intervals, and every object tentatively accepts some proposals based on priorities as well as the "first-come first-serve" principle. We first give a simple example to illustrate this procedure.

Example 2. *Suppose that $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. The preferences and priorities are given as follows.*

R_1	R_2	R_3		$\succeq_a$	$\succeq_b$	$\succeq_c$
b	a	a		1	$1, 2$	2
c	b	b		$2, 3$	3	1
a	c	c				3

In the first step, every agent proposes to consume her favorite object (at the unit rate) during the time interval $[0, 1]$. Agent 1's proposal is tentatively accepted by object b. Given that object a cannot accommodate the proposals of both agent 2 and agent 3, and $2 \sim_a 3$, its acceptance decision is based on the first-come first-serve principle. In this case, the two agents come at the same time $t = 0$. If they consume from $t = 0$, the object is exhausted at $t = \frac{1}{2}$. Therefore, object a tentatively accepts both agents' proposals to consume during $[0, \frac{1}{2}]$, and rejects their proposals to consume during $[\frac{1}{2}, 1]$.

Since they each have a (portion of) proposal rejected, in the second step, 2 and 3 propose to consume their second choice, object b, during $[\frac{1}{2}, 1]$. Recall that b has tentatively accepted 1's proposal to consume during $[0, 1]$ in the first step. Then the proposal of 3

[18] Specifically, under strict priorities, DA gives a stable (and ex-ante fair) deterministic allocation that Pareto dominates every other stable deterministic allocation. Since every ex-ante fair allocation can be represented as a lottery over some stable deterministic allocations, the outcome of DA Pareto dominates every other ex-ante fair allocation. On the other hand, in house allocation, there is a unique individually rational, non-wasteful and ordinally fair allocation (Hashimoto et al., 2014).

is first rejected as she has a lower priority than the other agents. If 1 and 2 consume b according to their proposals, on the first-come first-serve basis, then 1 starts to consume from $t = 0$, while 2 starts to consume from $t = \frac{1}{2}$. b is exhausted at $t = \frac{3}{4}$, and hence it tentatively accepts 1's proposal to consume during $[0, \frac{3}{4}]$, and 2's proposal to consume during $[\frac{1}{2}, \frac{3}{4}]$. In addition, it rejects their proposals to consume during $[\frac{3}{4}, 1]$.

In the third step, agents 1, 2, and 3 propose to consume object c during $[\frac{3}{4}, 1], [\frac{3}{4}, 1]$ and $[\frac{1}{2}, 1]$, respectively. These proposals are accepted, and the procedure terminates. The final consumption schedule gives the following allocation, which is agent-optimal ex-ante fair.

	a	b	c
1	0	$\frac{3}{4}$	$\frac{1}{4}$
2	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$
3	$\frac{1}{2}$	0	$\frac{1}{2}$

As can be seen in this example, the outcome allocation is ex-ante stable since an object always accepts proposals in favor of agents with higher priorities. On the other hand, ordinal fairness is guaranteed by the first-come first-serve principle.

We give the formal definition of the mechanism. Consider any problem $p = (N, A, R, \succeq)$. For an object $a \in A$, a *proposal* from an agent $i \in N$ is represented by an interval $[x, y]$, where $0 \leq x < y \leq 1$, meaning that agent i proposes to consume object a during $[x, y]$. Given a collection of proposals $\{[x_i, y_i]\}_{i \in S}$ from some agents $S \subseteq N$, which is referred to as a *choice set*, each object a has the following *choice rule*. If $\sum_{i \in S}(y_i - x_i) \leq 1$, then it chooses all the proposals; otherwise, it chooses the proposals $\{[x_i, y_i']\}_{i \in S'}$, where $S' \subseteq S$, that satisfy the following: $\sum_{i \in S'}(y_i' - x_i) = 1$, and there exists $i^* \in S'$ such that for any $j \in S$:

- If $j \succ_a i^*$, then $j \in S'$ and $y_j' = y_j$.
- If $j \sim_a i^*$ and $x_j < y_{i^*}'$, then $j \in S'$ and $y_j' = \min\{y_j, y_{i^*}'\}$.
- If $i^* \succ_a j$, or, $j \sim_a i^*$ and $x_j \geq y_{i^*}'$, then $j \notin S'$.

That is, when $\sum_{i \in S}(y_i - x_i) > 1$, object a's choice is based on multiple consumption processes, where the agents in a higher priority class consume before those in a lower priority class, and within the same priority class, each agent i starts to consume a from $t = x_i$. In the above definition, when the agents with the same priority as i^* consume a, it is exhausted at $t = y'_{i^*}$. Using these choice rules, we define the **deferred consumption procedure**:

> **Step 1.** Each agent proposes to consume her favorite option (which is an object or her outside option) during $[0, 1]$. Each object chooses from the proposals that it receives, tentatively accepts some proposals by its choice rule, and rejects the remaining proposals.
>
> **Step $k \geq 2$.** For each agent, if she has a proposal $[x, y]$ rejected by some object a in the last step, then she proposes to consume her next best option to object a during $[x, y]$.[19] Each object considers the proposals received in this step, as well as the proposals tentatively accepted earlier. Then, it tentatively accepts some proposals by its choice rule, and rejects the remaining proposals.
>
> The procedure terminates in some step k if no object rejects any proposal in this step.

An object a may need to consider two different proposals $[x^1, x^2]$ and $[x^2, x^3]$ from the same agent in some step. In this case we combine the proposals: let $[x^1, x^3]$ be the only proposal from this agent in the choice set of object a. Then we have the following useful fact that makes the procedure more transparent. It also indicates that the choice rule of each object can always be appropriately applied.

Observation 1. *Consider any step k of the deferred consumption procedure. After agents propose in this step, for each $i \in N$ there exist $x^1 < x^2 <, \ldots, < x^\ell$, where $x^1 = 0$ and $x^\ell = 1$, such that:*

- *For any $1 \leq \ell_1 \leq \ell - 1$, agent i's proposal $[x^{\ell_1}, x^{\ell_1+1}]$ is in the choice set of some $a \in A$, or she has proposed to consume her outside option during $[x^{\ell_1}, x^{\ell_1+1}]$.*[20]
- *If $1 \leq \ell_1 < \ell_2 \leq \ell - 1$, then her proposal $[x^{\ell_1}, x^{\ell_1+1}]$ is in the choice set of some $a \in A$, and either her proposal $[x^{\ell_2}, x^{\ell_2+1}]$ is in the choice set of some $b \in A$ with $aP_i b$, or she has proposed to consume her outside option during $[x^{\ell_2}, x^{\ell_2+1}]$.*

[19] She may also have a proposal $[z, w]$ rejected by another object, object b, in the last step, and hence in this step she also proposes to consume her next best option to b during $[z, w]$. That is, in general an agent may propose to multiple objects in a step.

[20] We also combine proposals to the outside option. For instance, if an agent proposes to consume her outside option during $[x^2, x^3]$ in a step, and proposes to consume her outside option during $[x^1, x^2]$ in a later step, we simply say that she has proposed to consume her outside option during $[x^1, x^3]$.

It is straightforward to prove this partition result by induction.[21] Given the numbers $x^1, \dots, x^\ell$ above, for each $a \in A$, let $M^k_{ia} = x^{\ell_1+1} - x^{\ell_1}$ if there exists $1 \leq \ell_1 \leq \ell - 1$ such that agent i's proposal $[x^{\ell_1}, x^{\ell_1+1}]$ is in the choice set of $a \in A$, and $M^k_{ia} = 0$ otherwise. We can interpret the $|N| \times |A|$ matrix M^k as the *tentative assignment* after agents propose in step k, which is not necessarily a well-defined random allocation. Let $M^k_{ii} = 1 - \sum_{a \in A} M^k_{ia}$ for each $i \in N$. In general, as in Example 3 below, the deferred consumption procedure may not be finite due to cycles in priorities, leading to an infinite sequence of tentative assignments $\{M^k\}_{k=1}^{\infty}$. For simplicity, if the procedure terminates in step $\bar{k}$, then we still construct an infinite sequence $\{M^k\}_{k=1}^{\infty}$ by setting $M^k = M^{\bar{k}}$ for all $k > \bar{k}$.

In the deferred consumption procedure, an agent always proposes to consume her next best option after she has a proposal rejected. This implies that, for all $i \in N$ and $a \in A$, $\left\{F(R_i, a, M^k)\right\}_{k=1}^{\infty}$ is a decreasing (and bounded) sequence, and hence it converges.[22] It follows that the sequence $\{M^k_{ia}\}_{k=1}^{\infty}$ also converges. Therefore, the outcome of the deferred consumption procedure for the current problem p is represented by a $|N| \times |A|$ matrix $f^{\mathrm{DC}}(p)$ such that

$$f^{\mathrm{DC}}_{ia}(p) = \lim_{k \to \infty} M^k_{ia}$$

for all $i \in N$ and $a \in A$.

Theorem 1. *For any problem p, $f^{DC}(p)$ is the agent-optimal ex-ante fair random allocation.*

All the proofs are given in Appendix A. We refer to f^{DC} as the **deferred consumption mechanism** (DC). Below we give another example to illustrate the mechanism, where the deferred consumption procedure is infinite.[23]

[21] It obviously holds for $k = 1$: in the first step, each agent's proposal $[0, 1]$ is in the choice set of her favorite object or she has proposed to consume her outside option during $[0, 1]$. Then, if it is true for some $k \geq 1$, it is also true for $k + 1$, which essentially follows from the fact that an agent always proposes to the next best option once she has a proposal rejected.

[22] We abused the notation slightly, since $F(R_i, a, M)$ was only defined for an allocation M. But this will not cause any confusion.

[23] Suppose that we take ex-ante fairness as constraints on allocations. In light of Theorem 1, the deferred consumption procedure helps establish the non-emptiness of the set of allocations satisfying the constraints, as well as the uniqueness of the constrained sd-efficient allocation. Therefore, the vigilant eating rule from Aziz and Brandl (2021), as mentioned in Section 1.1, gives an algorithm to find the agent-optimal ex-ante fair allocation in a finite number of steps (through a procedure that is inherently different from deferred consumption).

Example 3. *Suppose that $N = \{1,2,3\}$ and $A = \{a,b,c\}$. The preferences and priorities are given as follows.*

R_1	R_2	R_3
b	b	a
c	a	b
a	c	c

$\succeq_a$	$\succeq_b$	$\succeq_c$
1	3	2
2,3	1,2	1
		3

First, we have $M^1_{1b} = M^1_{2b} = M^1_{3a} = 1$. Then, for every even number $k \geq 2$, the tentative assignments M^k and M^{k+1} are given by the following matrices:

	a	b	c
1	0	$\frac{1}{2}x_k$	$1-\frac{1}{2}x_k$
2	$\frac{1}{2}x_k$	$\frac{1}{2}x_k$	$1-x_k$
3	x_k	$1-x_k$	0

M^k

	a	b	c
1	0	$\frac{1}{2}x_k$	$1-\frac{1}{2}x_k$
2	$\frac{1}{2}-\frac{1}{4}x_k$	$\frac{1}{2}x_k$	$\frac{1}{2}-\frac{1}{4}x_k$
3	$\frac{1}{2}+\frac{1}{4}x_k$	$\frac{1}{2}-\frac{1}{4}x_k$	0

M^{k+1}

where $x_2 = 1$, and if $k \geq 4$,

$$x_k = 1 - \sum_{\ell=1}^{\frac{1}{2}k-1} (\frac{1}{4})^\ell.$$

For any even k, object a is over-assigned in M^k (i.e., $\sum_{i\in N} M^k_{ia} > 1$),[24] and in step k it rejects a proposal with some measure $y > 0$ from both agent 2 and agent 3. Then, 3 proposes to consume object b in step $k+1$, which leads to both 1 and 2 having a proposal with measure $\frac{1}{2}y$ rejected by b. In step $k+2$, agent 2 proposes to consume object a again, which then rejects a proposal with measure $\frac{1}{4}y$ from both 2 and 3. The procedure continues in this way infinitely, and the tentative assignments converge to the following agent-optimal ex-ante fair allocation:

	a	b	c
1	0	$\frac{1}{3}$	$\frac{2}{3}$
2	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
3	$\frac{2}{3}$	$\frac{1}{3}$	0

Due to the specification of the objects' choice rules, the deferred consumption procedure eventually produces a consumption schedule in which, at any point of time, every

[24]Note that $\frac{2}{3} < x_k \leq 1$ for any even k.

agent does not envy the object being consumed by another agent, unless the latter agent has a higher priority for the object. In the special case of house allocation, such consumption schedule can be obtained by simply letting the agents simultaneously consume the best available objects. This gives an intuitive explanation on why DC is equivalent to PS in this case. On the other hand, it is obvious from the construction that DC is reduced to DA in the special case of strict priorities.

Regarding incentive compatibility, DC is not a strategy-proof mechanism. For instance, comparing the agent-optimal ex-ante fair allocations in Examples 1 and 3, agent 3 can manipulate to receive her first choice with a larger probability. It is also already known that PS is not strategy-proof (Bogomolnaia and Moulin, 2001). Therefore, a strategy-proof and ex-ante fair mechanism generally does not exist.

5 Properties of Ex-Ante Fair Allocations

In this section, we present the interesting properties of ex-ante fair allocations, beyond the fact that there exists an agent-optimal one. In particular, it will be shown that, as suggested by Example 1, the (potentially infinite) set of ex-ante fair allocations is a complete lattice under the common preferences of the agents. We will also explicitly construct the least upper bound and the greatest lower bound of any subset of ex-ante fair allocations, which have natural interpretations.

Fix a problem $p = (N, A, R, \succeq)$. Denote the set of all ex-ante fair allocations as $\mathscr{E}$, and consider the partially ordered set $(\mathscr{E}, R_N^{sd})$. For any non-empty $S \subseteq \mathscr{E}$, we define two $|N| \times |A|$ matrices, $\vee S$ and $\wedge S$, as follows: for every $i \in N$ and $a \in A$,

$$(\vee S)_{ia} = \sup\, \{F(R_i, a, M) : M \in S\} - \sup\, \{F(P_i, a, M) : M \in S\}, \text{ and}$$

$$(\wedge S)_{ia} = \inf\, \{F(R_i, a, M) : M \in S\} - \inf\, \{F(P_i, a, M) : M \in S\}.$$

Moreover, let $(\vee S)_{ii} = 1 - \sum_{b \in A} (\vee S)_{ib}$ and $(\wedge S)_{ii} = 1 - \sum_{b \in A} (\wedge S)_{ib}$. The following result indicates that if $\vee S$ and $\wedge S$ are ex-ante fair allocations, then they are the least upper bound and the greatest lower bound of S under R_N^{sd}, respectively.

Lemma 1. *For any $i \in N$ and $a \in A \cup \{i\}$, we have $F(R_i, a, \vee S) = sup\, \{F(R_i, a, M) : M \in S\}$, and $F(R_i, a, \wedge S) = inf\, \{F(R_i, a, M) : M \in S\}$.*

The next theorem establishes the lattice structure as well as other key properties of ex-ante fair allocations.

Theorem 2. *If $\{M, M'\} \subseteq S \subseteq \mathscr{E}$, then we have the following:*

(i) *(Complete lattice) $\vee S \in \mathscr{E}$ and $\wedge S \in \mathscr{E}$. Moreover, for any $i \in N$, either $M_i R_i^{sd} M_i'$ or $M_i' R_i^{sd} M_i$; for any $a \in A$, either $M_a \succeq_a^{sd} M_a'$ or $M_a' \succeq_a^{sd} M_a$.*

(ii) *("Rural hospital" theorem) For any $i \in N$ and $a \in A$, $\sum_{b\in A} M_{ib} = \sum_{b\in A} M_{ib}'$, and $\sum_{j\in N} M_{ja} = \sum_{j\in N} M_{ja}'$. Moreover, $M_i = M_i'$ if $\sum_{b\in A} M_{ib} < 1$, and $M_a = M_a'$ if $\sum_{j\in N} M_{ja} < 1$.*

(iii) *(Two-sided efficiency) There does not exist a random allocation $M'' \neq M$ such that $M'' R_N^{sd} M$ and $M'' \succeq_A^{sd} M$.*

(iv) *(Conflicting interests) $M R_N^{sd} M'$ if and only if $M' \succeq_A^{sd} M$.*

If the priorities are strict, then, for deterministic allocations, ex-ante fairness is equivalent to stability. Therefore, the above results generalize the familiar properties of stable matchings in the classical two-sided matching market, where both sides of the market have strict preferences, i.e., the *marriage problem* considered in Gale and Shapley (1962).[25]

The first part of statement (i) in Theorem 2 extends Conway's lattice theorem of stable matchings (Knuth, 1976). Since $(\mathscr{E}, R_N^{sd})$ is a complete lattice, there exists a maximum ex-ante fair allocation, $\vee\mathscr{E}$, which is the outcome of DC, as well as a minimum ex-ante fair allocation, $\wedge\mathscr{E}$. Although $\succeq_A^{sd}$ is generally not antisymmetric on all the random allocations due to the priorities being weak, statement (iv) in Theorem 2 implies that it is a partial order on the set $\mathscr{E}$, and $(\mathscr{E}, \succeq_A^{sd})$ is also a complete lattice.

The second part of statement (i) says that every agent and object can compare the lotteries under two ex-ante fair allocations using the first-order stochastic dominance relation, which sheds more light on the lattice structure: if S is finite, then $\vee S$ (resp. $\wedge S$) can be constructed by letting each agent pick the best (resp. the worst) lottery from the ones under the allocations in S. However, the least upper bound or the greatest lower bound of S under $\succeq_A^{sd}$ cannot be easily obtained by letting objects pick lotteries: although $\succeq_A^{sd}$ is antisymmetric on $\mathscr{E}$ as discussed above, for an individual object $a \in A$, $\succeq_a^{sd}$ may not be antisymmetric on the set $\{M_a : M \in \mathscr{E}\}$.[26]

[25]These properties of stable matchings, as well as Theorem 2, can also be easily extended to the many-to-one setting. See Roth and Sotomayor (1990) for detailed discussions of the two-sided matching market.

[26]For instance, in Example 1, for every $x, y \in (\frac{1}{3}, \frac{1}{2}]$ such that $x \neq y$, we have $M_a(x) \neq M_a(y)$, $M_a(x) \succeq_a^{sd} M_a(y)$ and $M_a(y) \succeq_a^{sd} M_a(x)$.

Besides the lattice structure, the *rural hospital theorem* is another fundamental property of stable matchings in the two-sided matching market, which is established by McVitie and Wilson (1970) for the case of one-to-one matching, and Roth (1984, 1986) for the case of many-to-one matching. Statement (ii) in Theorem 2 shows a probabilistic version of this result for ex-ante fair allocations: each object's probability of being assigned to some agent is constant among $\mathscr{E}$, and if this probability is less than one, then the object receives the same lottery under every ex-ante fair allocation. An analogous result also holds for each agent.

The last two statements in Theorem 2 have better economic interpretations if we envision that each object has intrinsic preferences that are aligned with its priority ordering. That is, each object always prefers an agent with a higher priority. In this context, statement (iii) says that every ex-ante fair allocation is "sd-efficient" for the two sides of the market, i.e., when the welfare of every agent and object is taken into account.[27] Finally, statement (iv) shows that the agents and the objects have conflicting interests regarding ex-ante fair allocations: every agent weakly prefers M to M', if and only if every object weakly prefers M' to M.

In addition, for the special case of house allocation, statement (iii) implies that an ex-ante fair allocation, or equivalently, an individually rational, non-wasteful and ordinally fair allocation, is sd-efficient.[28] Moreover, by the lattice structure, such allocation is unique. On the other hand, it is straightforward to check that PS is individually rational, non-wasteful and ordinally fair. Therefore, we obtain the first characterization result of PS in Hashimoto et al. (2014).

Under weak priorities, the results in Theorem 2 generally do not hold for stable deterministic allocations (or any other concept that we are aware of). For instance, the set of stable deterministic allocations may not be a lattice, and there may not exist a stable deterministic allocation that Pareto dominates every other stable deterministic allocation.

[27] Note that even the agent-optimal ex-ante fair allocation may not be sd-efficient (for the agents). It is well-known that a stable and efficient deterministic allocation may not exist (Roth, 1982, Abdulkadiroğlu and Sönmez, 2003b). Hence, in general, ex-ante fairness and sd-efficiency are not compatible.

[28] This is because if M is an ex-ante fair allocation, M'' is some random allocation, and $M'' R_N^{sd} M$, then by Lemma 3 in Appendix A (the *Reshuffling Lemma* from Erdil (2014)), $\sum_{i\in N} M_{ia} = \sum_{i\in N} M''_{ia}$ for all $a \in A$, and hence $M'' \succeq_A^{sd} M$.

6 Division

To better understand the concept of ex-ante fairness, we propose and discuss a new method of generating random allocations from deterministic allocations, which we refer to as *division*. This offers a new perspective on some of the key results in Theorem 2, as well as the construction of DC.

We start with the simplest example to illustrate the idea of division. There is only one object, a, to be allocated to two agents, i and j, who both desire it and have equal claim to it, i.e., $i \sim_a j$. Obviously, by any standard the only fair random allocation is $M_{ia} = M_{ja} = 0.5$. We take the stance that deterministic allocations are more fundamental than random allocations, and want to understand how the random allocation M can be constructed from deterministic allocations. The first interpretation is that M is generated by the method of *randomization*. That is, we randomize over the two deterministic allocations in which one agent receives a, such that each of the two is picked with probability 0.5.

We next give another possible interpretation. To resolve the conflicting claims of the two agents, we divide the claim of each $k \in \{i, j\}$ into two parts, k^1 and k^2. The object is also divided into two parts, a^1 and a^2, so that each k^x, where $k \in \{i, j\}$ and $x \in \{1, 2\}$, represents a claim to one part of the object. We then prioritize the divided claims: for both parts of the object, i^1 and j^1 have the same priority, i^2 and j^2 have the same priority, and each of i^1 and j^1 has a higher priority than each of i^2 and j^2. This leads to a *divided problem*, in which we essentially allocate two objects, $\{a^1, a^2\}$, to four agents, $\{i^1, i^2, j^1, j^2\}$, where each agent finds the two objects indifferent. This division operation treats the two agents in the same way, and a fair deterministic allocation can be found in the divided problem due to the more refined priority structure: there exists a (strongly) stable deterministic allocation where i^1 and j^1 each receives one object, which generates the random allocation M for the original problem.

This idea of division can be easily extended to an arbitrary problem. In general, we divide (the claim of) each agent, as well as each object, into some finite number of parts. A part of an agent i has weak preferences over the objects parts, which are simply extended from the preferences of i. Moreover, we grant different priorities to different parts of each agent. Given the priorities in the original problem, the parts of a higher ranked agent are always ranked higher, and, for equally ranked agents, their parts are ranked based on indices.

We provide a formal framework that will also be used throughout Section 7. Consider a problem $p = (N, A, R, \succeq)$. For simplicity, assume that $|N| = |A|$, $aP_i i$ for all $a \in A$ and $i \in N$, and every allocation is a bistochastic matrix.[29] Given an integer $q > 0$, each agent $i \in N$ is divided into q parts, $i^1, \ldots, i^q$. Let $N^q = \{i^x : i \in N, x = 1, \ldots, q\}$. Each object $a \in A$ is also divided into q parts, $a^1, \ldots, a^q$, and let $A^q = \{a^x : a \in A, x = 1, \ldots, q\}$. Each $i^x \in N^q$ has a preference relation R_i^q on A^q such that for all $a^y, b^z \in A^q$, $a^y R_i^q b^z$ if and only if $aR_i b$. Then P_i^q and I_i^q denote the asymmetric and symmetric components of R_i^q, respectively. Each $a^x \in A^q$ has a priority ordering $\succeq_a^q$ over N^q such that for all $i^y, j^z \in N^q$, $i^y \succeq_a^q j^z$ if and only if either $i \succ_a j$, or, $i \sim_a j$ and $y \leq z$. Let $\succ_a^q$ and $\sim_a^q$ denote the asymmetric and symmetric components of $\succeq_a^q$, respectively. Then, $p^q = (N^q, A^q, R^q, \succeq^q)$ denotes the ***q*-divided problem** of p.

We assume the preferences are strict in the original problem p as before, although p^q is a problem with weak preferences. Alternatively, p^q can be interpreted as a many-to-one problem with strict preferences, as each part of an agent finds all parts of an object indifferent. A deterministic allocation μ for p^q generates a random allocation $M(\mu, p^q)$ for p: for all $i \in N$ and $a \in A$,

$$M_{ia}(\mu, p^q) = \frac{1}{q}\Big|\big\{x \in \{1, \ldots, q\} : \mu(i^x) = a^y \text{ for some } y \in \{1, \ldots, q\}\big\}\Big|.$$

Compared with the randomization method, the division method could deliver better efficiency and fairness properties from the ex-ante perspective. To start with, it is straightforward to see that if μ is efficient for p^q, then $M(\mu, p^q)$ is sd-efficient for p. In contrast, a randomization over efficient deterministic allocations gives a random allocation that usually only satisfies the weaker notion of *ex-post efficiency*.[30] Regarding fairness, in general, we can choose a deterministic allocation μ that respects the more refined priority structure in the divided problem in some way, such that $M(\mu, p^q)$ satisfies some desirable fairness properties.[31] As one example, while a randomization over stable deterministic allocations only leads to an ex-post stable random allocation, $M(\mu, p^q)$ is ex-ante stable if μ is stable. More importantly, it can also be easily shown that $M(\mu, p^q)$ is ex-ante fair if μ is strongly stable.

[29] Under these assumptions, we no longer need to consider individual rationality or non-wastefulness. Moreover, for each $i \in N$, we only need to specify her preferences over A (instead of $A \cup \{i\}$).

[30] Formally, a random allocation is ex-post efficient if it can be represented as a lottery over efficient deterministic allocations.

[31] In applications, we do not necessarily respect the priority structure in the sense of stability. See the random mechanisms discussed in Sections 7.2 and 7.3.

Therefore, strongly stable deterministic allocations can generate ex-ante fair random allocations through the division method. But in general we cannot construct the whole set of ex-ante fair allocations in this way. For instance, in Example 1, there is a continuum of ex-ante fair allocations, and hence there does not exist an integer q such that every one of them is generated by a strongly stable deterministic allocation in the q-divided problem. Motivated by this issue, we envision a *continuum divided problem*, where each agent and object $o \in N \cup A$ is divided into a continuum of parts with measure 1, represented by $\{o^x : x \in [0,1]\}$, and preferences and priorities are defined as before.

For every ex-ante fair allocation M, we can find a bijection $\mu : \{i^x : i \in N, x \in [0,1]\} \rightarrow \{a^x : a \in A, x \in [0,1]\}$ such that for any $i \in N$ and $x \in [0,1]$, $\mu(i^x) \in \{a^y : y \in [0,1]\}$ if $F(P_i, a, M) < x \leq F(R_i, a, M)$, or, $F(P_i, a, M) = 0 = x < F(R_i, a, M)$. Then μ is a strongly stable deterministic allocation for the continuum divided problem, which generates M for the original problem. Note that the priorities in the continuum divided problem are almost strict, and hence strong stability is almost equivalent to stability. Therefore, such connection between ex-ante fairness and (strong) stability helps explain why the lattice theorem as well as the rural hospital theorem can be extended to ex-ante fair allocations.

More specifically, it is possible to prove the lattice structure of ex-ante fair allocations, "for any $a \in A$, either $M_a \succeq_a^{sd} M'_a$ or $M'_a \succeq_a^{sd} M_a$" in statement (i), and statement (ii) in Theorem 2 (except "$M_i = M'_i$ if $\sum_{b \in A} M_{ib} < 1$"), by extending the proofs of the existing results in the two-sided matching market to the continuum divided problem. However, we cannot do the same for the other results in Theorem 2. On one hand, the completeness of the lattice, "for any $i \in N$, either $M_i R_i^{sd} M'_i$ or $M'_i R_i^{sd} M_i$" in statement (i), and "$M_i = M'_i$ if $\sum_{b \in A} M_{ib} < 1$" in statement (ii) cannot be explained by existing properties of stable matchings. On the other hand, the two-sided efficiency and the conflicting interests result of stable matchings do not directly translate to statements (iii) and (iv).[32] Therefore, in the proof of Theorem 2, we show all the results directly, without regard to any divided problem, which we believe is a more natural and straightforward approach.

In our study, the continuum divided problem only serves as a conceptual tool. It helps better understand ex-ante fairness and some of its key properties, and contributes to the idea behind the construction of DC. This mechanism is essentially generated by applying DA to continuum divided problems: when agent i's parts $\{i^x : x \in [y,z]\}$

[32] One key reason is that if M is ex-ante fair and $M'_a \succeq_a^{sd} M_a$, then in the continuum divided problem, the deterministic allocation generating M' is not necessarily "better" for a than the one generating M, i.e., the parts of a may be assigned to higher ranked parts of agents under the latter.

apply to the parts of object a, this is interpreted as i's proposal to consume a during the time interval $[y,z]$. However, instead of tracking the atomless parts of agents in a continuum divided problem, in Section 4 we focused on the original problem directly, and gave a simple and intuitive description of the mechanism.

While applying DA to continuum divided problems does not generate a strategy-proof mechanism, DA with single tie-breaking, a randomization over DA, is strategy-proof. In general, there is a clear trade-off between the randomization method and the division method. Although we can achieve better ex-ante properties of fairness and efficiency using the division method, applying a strategy-proof deterministic mechanism to divided problems usually does not generate a strategy-proof random mechanism, which will also be seen in other examples in the next section. In contrast, a randomization over strategy-proof deterministic mechanisms is strategy-proof (as long as the distribution is preference-independent).

7 Additional Applications of Division

Finally, we present some additional applications of the division method, and mostly focus on allocation problems with simple and special priority structures. Generalizations of PS have been proposed to these problems, and we show that, similar to our generalization of PS, they can also be interpreted as random mechanisms generated by the division method. Specifically, they can be obtained by applying well-known deterministic mechanisms to finitely divided problems. In Sections 7.1 and 7.3, we also discuss the possibility of discovering additional new random mechanisms using our method.

For ease of exposition, in this section, assume that for any problem $p=(N,A,R,\succeq)$ under consideration, $|N|=|A|$, $aP_i i$ for all $a\in A$ and $i\in N$, and every allocation is a bistochastic matrix.

7.1 House Allocation Under Weak Preferences

In a house allocation problem under weak preferences, $p=(N,A,R,\succeq)$, R_i is not necessarily antisymmetric for any $i\in N$. Moreover, $i\sim_a j$ for all $i,j\in N$ and $a\in A$. Let $\mathscr{P}_{\text{HA}}$ denote the collection of all such problems.

Given any problem $p=(N,A,R,\succeq)$ under weak preferences, we say two allocations M and M' are *welfare equivalent* if $MR_N^{sd}M'$ and $M'R_N^{sd}M$. Moreover, for any $i\in N'\subseteq N$

and $A' \subseteq A$, let $B_i(A')$ denote the set of maximal elements in A' according to R_i, and $B_{N'}(A') = \cup_{j \in N'} B_j(A')$.

Katta and Sethuraman (2006) generalize PS to allow for weak preferences by introducing the **extended PS** (EPS) solution. For any $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HA}}$, EPS is defined as follows.[33]

> Let $A_0 = A$, $E_0 = \emptyset$, and $d_0(i) = 0$ for all $i \in N$. In each step $k \geq 1$, let $A_k = A_{k-1} \setminus E_{k-1}$, and N_k be the largest set of agents that solves the following problem:
> $$\min_{N' \subseteq N, N' \neq \emptyset} \frac{|B_{N'}(A_k)| - \sum_{i \in N'} d_{k-1}(i)}{|N'|}$$
> Define
> $$\lambda_k = \frac{|B_{N_k}(A_k)| - \sum_{i \in N_k} d_{k-1}(i)}{|N_k|}$$
> and $E_k = B_{N_k}(A_k)$. Then the objects E_k are allocated to N_k in step k: each $i \in N_k$ is assigned $\lambda_k + d_{k-1}(i)$ of the objects in $B_i(A_k)$. Set $d_k(i) = 0$ if $i \in N_k$, and $d_k(i) = d_{k-1}(i) + \lambda_k$ otherwise. The procedure terminates in step $\bar{k}$ if $E_{\bar{k}} = A_{\bar{k}}$.

In each step k, the "bottleneck set" N_k is identified. There may be multiple ways of allocating the objects E_k to the agents N_k, but all the outcome allocations are welfare equivalent. Denote the set of these allocations as $f^{\text{EPS}}(p)$. Heo and Yılmaz (2015) show that an allocation M is ordinally fair if and only if $M \in f^{\text{EPS}}(p)$. Ordinal fairness further implies sd-efficiency, and *sd-envy-freeness*. The latter is a stronger notion than equal treatment of equals: an allocation M is **sd-envy-free** if $M_i R_i^{sd} M_j$ for all $i, j \in N$.

Next, we relate EPS to the classical efficient deterministic mechanisms for $\mathscr{P}_{\text{HA}}$: **serial dictatorships** from Svensson (1994). Given any problem $p = (N, A, R, \succeq)$ under weak preferences, choose an ordering σ of the agents, where $\sigma : \{1, \ldots, |N|\} \to N$ is a bijection. Let D_0 be the set of all deterministic allocations for p. For $k \geq 1$, define D_k as the set of deterministic allocations most preferred by $\sigma(k)$ among the ones in D_{k-1}. That is,

$$D_k = \left\{\mu \in D_{k-1} : \mu(\sigma(k)) R_{\sigma(k)} \varphi(\sigma(k)) \text{ for all } \varphi \in D_{k-1}\right\}.$$

[33] We adopt the simplified definition from Heo and Yılmaz (2015), which is sufficient for our purpose. For the general idea behind the construction, see the original definition using networks in Katta and Sethuraman (2006).

Then $f^{\text{SD}}(\sigma, p) = D_{|N|}$ is the set of (welfare equivalent) deterministic allocations selected by the serial dictatorship with respect to σ.

For any $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HA}}$ and integer $q > 0$, let $\mathscr{O}(p, q)$ be the collection of orderings of N^q such that for any $\sigma \in \mathscr{O}(p, q)$ and $i^x, j^y \in N^q$ with $x < y$, $\sigma^{-1}(i^x) < \sigma^{-1}(j^y)$. We are ready to present the main result in this subsection.

Proposition 1. *Consider any $p = (N, A, R, \succeq) \in \mathscr{P}_{HA}$. Let $q = (n!)^n$, where $n = |N| = |A|$. Then for any $\sigma \in \mathscr{O}(p, q)$ and $\mu \in f^{SD}(\sigma, p^q)$, $M(\mu, p^q) \in f^{EPS}(p)$.*

Thus, applying serial dictatorships to the divided problems generates EPS. The ordering σ has to be chosen from the set $\mathscr{O}(p, q)$ to respect the differences in the priorities in p^q. After dividing each agent and object into the "correct" number of parts, i.e., $q = (n!)^n$, for every $x \in \{1, \ldots, q\}$, the ordering among the agents in $\{i^x : i \in N\}$ does not affect the outcome of the serial dictatorship in terms of welfare, and none of them envies another's assignment. Therefore, each $\mu \in f^{\text{SD}}(\sigma, p^q)$ is strongly stable for p^q.

If we want to fairly allocate 1 object among n agents by division, the object has to be divided into a multiple of n parts. The choice of q in the above proposition can be explained by extending this simple intuition. Consider the allocation procedure in a serial dictatorship. For simplicity, assume that the preferences R are strict, and envision that the object parts A^q are sequentially allocated to the agents N (instead of the agent parts N^q) such that each agent receives q object parts in the end. The parts of each object are at first allocated among the agents who consider it as the best choice, so it needs to be divided into a multiple of $n!$ parts to accommodate all possible numbers of such agents. An agent may start to choose the parts of a different object due to the exhaustion of objects, and hence the remaining parts of an object may later be allocated among a different set of agents. Then the fair allocation of these remaining parts requires q to be a multiple of $(n!)^2$. Thus, we choose $q = (n!)^n$, where the power of $n!$ reflects, for each object, the maximal number of times that the set of agents who choose this object changes.[34] More generally, if k objects are allocated to n agents, then we choose $q = (n!)^k$.[35]

[34] For the other similar results in the rest of Section 7, the choice of q follows the same principle.

[35] It is worth noting that this number also appears in the randomization method: if a deterministic mechanism is applied after an ordering of the agents is randomly and independently chosen for each object to break the ties in its priorities, then the resulting allocation is a randomization over $(n!)^k$ deterministic allocations. A familiar example of such random mechanism is deferred acceptance with *multiple tie-breaking* (Abdulkadiroğlu et al., 2009).

A key result in Kesten (2009) embeds the idea of division. He considers a house allocation problem p under strict preferences, and replicates each object such that there are k copies of it. For each ordering of the agents, let the agents choose objects sequentially. They choose one object at a time so that the serial dictatorship is applied k times. Then a random allocation is computed by taking the average of the $k(n!)$ possible serial dictatorship outcomes. He shows that this random allocation converges to the PS outcome as $k \to \infty$. Technically, this replication method is essentially equivalent to division. In our context, (the proof of) his result implies that for any $\sigma \in \mathcal{O}(p,q)$, as $q \to \infty$, $M(f^{\text{SD}}(\sigma, p^q), p^q)$ converges to the PS outcome. While the focus of Kesten (2009) is on explaining the efficiency loss of random serial dictatorship, we emphasize the idea of generating random mechanisms through division and the importance of choosing the correct division number, in a more general setting with weak preferences.

The nature of the original construction of EPS through networks is rather complicated. Our approach provides a more transparent construction and better understanding of this solution, by relating it to well-studied deterministic mechanisms.[36] Finally, we show that this approach can potentially produce new random mechanisms for $\mathscr{P}_{\text{HA}}$. Recall that, for any problem under strict preferences, a strongly stable deterministic allocation for its q-divided problem generates an ex-ante fair random allocation (see Section 6). As shown in the following example, this result no longer holds under weak preferences.

Example 4. *Let $p = (N, A, R, \succeq) \in \mathscr{P}_{HA}$, where $N = \{i, j, k, \ell\}$ and $A = \{a, b, c, d\}$. The preferences are given as follows.*

R_i	R_j	R_k	R_ℓ
a, b	a	d	d
c	c	b	c
d	b	c	a
	d	a	b

Let $q = 2$. The following deterministic allocation μ for p^q is strongly stable (and efficient):

$$\mu(i^1) = a^1, \mu(i^2) = b^1, \mu(j^1) = a^2, \mu(j^2) = c^1, \mu(k^1) = d^1, \mu(k^2) = b^2, \mu(\ell^1) = d^2, \mu(\ell^2) = c^2.$$

Below we give the allocation $M = M(\mu, p^q)$, as well as an allocation $M' \in f^{EPS}(p)$ for comparison.

[36] Also see Bogomolnaia (2015) for another reinterpretation of EPS, from a welfarist perspective.

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline i & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ j & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ k & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \ell & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array} \qquad \begin{array}{c|cccc} & a & b & c & d \\ \hline i & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ j & \frac{5}{6} & 0 & \frac{1}{6} & 0 \\ k & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ \ell & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

$$M \qquad\qquad M'$$

Note that M is not ordinally fair: $M_{ia} > 0$ but $F(R_i, a, M) = 1 > \frac{1}{2} = F(R_j, a, M)$.

In this example, although M is not ordinally fair and $M \notin f^{\text{EPS}}(p)$, M is sd-efficient and sd-envy-free. In general, it can be easily shown that for any $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HA}}$ and q, if μ is strongly stable for p^q, then $M(\mu, p^q)$ is sd-envy-free for p. Therefore, when sd-efficiency and sd-envy-freeness are the main desiderata, we can apply any efficient and strongly stable deterministic mechanism to divided problems to construct a desirable random mechanism (that is not necessarily EPS). Obviously, this also indicates that the connection between serial dictatorships and EPS established in Proposition 1 is not simply due to the efficiency and strong stability of serial dictatorships.

7.2 House Allocation with Existing Tenants

Next, we consider the situation in which some objects are private endowments while others are common endowments, or *house allocation with existing tenants* (Abdulkadiroğlu and Sönmez, 1999).[37] This can be modeled via weak priorities such that for each privately owned object the owner has a higher priority than all the other agents. Formally, let $\mathscr{P}_{\text{HET}}$ denote the collection of house allocation problems with existing tenants (under strict preferences). For each $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HET}}$, there exist non-empty $N(p) \subseteq N$ (the existing tenants), $A(p) \subseteq A$, and a bijection $e_p : N(p) \to A(p)$ such that

- for each $a \in A(p)$, $e_p^{-1}(a) \succ_a i \sim_a j$ for all $i, j \in N \setminus \{e_p^{-1}(a)\}$, and
- for each $a \in A \setminus A(p), i \sim_a j$ for all $i, j \in N$.

In such a problem, ex-ante stability is equivalent to the key requirement that each existing tenant's lottery first-order stochastically dominates the degenerate lottery of receiving her endowment, which guarantees voluntary participation. DC can be applied to

[37] A special case is a *housing market* (Shapley and Scarf, 1974), where each agent privately owns one object.

$\mathscr{P}_{\text{HET}}$, but it is not sd-efficient. Under the special priority structure in each $p \in \mathscr{P}_{\text{HET}}$, an ex-ante stable and sd-efficient allocation always exists.[38] Thus, in this context it is more appropriate to give up ordinal fairness, and look for an ex-ante stable and sd-efficient random mechanism that also respects the indifferences in priorities in some way. Both Yılmaz (2010) and Zhang (2017) propose such a mechanism by generalizing PS.

First, we refer to the mechanism from Yılmaz (2010) as **ex-ante stable PS**. Consider $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HET}}$. Under this mechanism, the agents still consume objects at the unit rate, but to satisfy the ex-ante stability constraints, for any $N' \subseteq N(p)$, the agents N' are entitled to the objects $U_{N'} = \cup_{i \in N'} \{a \in A : aR_i e_p(i)\}$ in the following sense: if at some point of time, the total remaining fractions of these objects are equal to the total remaining demands of N', then any agent not in N' cannot further consume the objects $U_{N'}$.[39]

One of the simplest stable and efficient deterministic solutions to the problem $p \in \mathscr{P}_{\text{HET}}$ is a *stable serial dictatorship*.[40] It is in the same vein as the serial dictatorships defined in Section 7.1, and the only difference is that the procedure starts with all stable deterministic allocations for p, i.e., the initial set D_0 is the set of deterministic allocations in which no existing tenant receives an object worse than her endowment. Then, we extend this idea of stable serial dictatorships to a divided problem p^q as follows. Let $\sigma \in \mathscr{O}(p, q)$. Define the initial set D_0 as the set of deterministic allocations (for p^q) in which no existing tenant has a part that is assigned a part of an object worse than her endowment. Then, for $k \geq 1$, let D_k be the set of deterministic allocations most preferred by $\sigma(k)$ among the ones in D_{k-1}. If $q = (n!)^{n^2}$, where $n = |N| = |A|$, it can be shown that for any $\mu \in D_{nq}$, $M(\mu, p^q)$ is the outcome of the ex-ante stable PS.[41]

Second, Zhang (2017) proposes the "**You request my house-I get your rate**" mechanism. In each step, if some existing tenants demand each other's endowment and form a cycle, then they exchange probability shares of their endowments. If there is no such cycle, then agents consume their best available objects at possibly different rates: each agent has the same *basic consuming rate* of 1, and she also gets an additional rate equal

[38]In fact, this is one of the very few types of priority structures for which (ex-ante) stability and (sd-) efficiency are compatible. See Han (2018) for a characterization of weak priority structures.

[39]That is, starting from this point of time, N' and the remaining fractions of $U_{N'}$ become an independent subproblem. Another subproblem may emerge within this one later, and hence the mechanism is defined recursively.

[40]See Biró et al. (2021) and Manjunath and Westkamp (2021) for discussions of such mechanisms in the context of multi-unit exchange of indivisible objects.

[41]We omit a formal proof, which uses very similar techniques as the proof of Proposition 1.

to the sum of the rates of those agents who are consuming her endowment.[42] Due to its trading feature, we also refer to this mechanism as *PS with trading* (PST) for simplicity, denoted as f^{PST}.

We relate PST to the original deterministic mechanisms proposed for $\mathscr{P}_{\text{HET}}$ in Abdulkadiroğlu and Sönmez (1999), the **top trading cycles** (TTC) mechanisms.[43] In fact, we need the more general form of the TTC mechanisms in Abdulkadiroğlu and Sönmez (2003b), defined for any priority-augmented allocation problem $p = (N, A, R, \succeq)$. Given an ordering σ of the agents N, the mechanism is defined through the following procedure.

> In each step $k \geq 1$, consider the remaining agents and the remaining objects in the *subproblem* p_k, where $p_1 = p$. Let each agent point to her favorite object, and each object point to the agent with the highest priority (if there is a tie, break it using σ). Then there exists at least one cycle, which we refer to as a *trading cycle*. Let each agent in a trading cycle be assigned the object that she points to. After removing the agents and objects in all trading cycles from p_k, the reduced subproblem is p_{k+1}. The procedure terminates when all the agents are assigned.

Denote the resulting deterministic allocation as $f^{\text{TTC}}(\sigma, p)$.

For TTC to generate PST through the division method, we need to make two slight modifications to the divided problems. Consider any $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HET}}$ and integer $q > 0$. As TTC is defined for strict preferences, we first break the ties in preferences: for each $i^x \in N^q$, define $\tilde{R}_i^q$ such that $a^y \tilde{R}_i^q b^z$ if and only if aP_ib, or, $a = b$ and $y \leq z$. Then TTC can be applied to $(N^q, A^q, \tilde{R}^q, \succeq^q)$. However, this may cause an existing tenant to lose her basic consuming rate. Therefore, we give a more favorable treatment to the existing tenants, by reversing the priority ordering of the parts of each privately owned object over the parts of its owner: for each $a^x \in A^q$, define $\tilde{\succeq}_a^q$ such that for all $i^y, j^z \in N^q$, if $a \in A(p)$ and $i = j = e_p^{-1}(a)$, then $i^y \tilde{\succeq}_a^q j^z \iff y \geq z$; otherwise, $i^y \tilde{\succeq}_a^q j^z \iff i^y \succeq_a^q j^z$. Then, the modified q-divided problem is denoted as $\tilde{p}^q = (N^q, A^q, \tilde{R}^q, \tilde{\succeq}^q)$.

[42] A formal definition of this mechanism is given in Appendix A.5.

[43] Zhang (2017) shows that PST can be interpreted as a procedure in which agents trade their *fractional endowments*, where in each step the remaining fractions of an object that is not owned by any remaining agent are equally distributed as private fractional endowments (in addition, Yu and Zhang (2020) define a more general parametric class of trading algorithms that incorporate PST as a special case). We will also interpret PST as a trading procedure, but from a different perspective. Our focus is on establishing an explicit connection between PST and the deterministic TTC mechanisms, through the method of division.

Proposition 2. *Consider any* $p = (N, A, R, \succeq) \in \mathscr{P}_{HET}$, *where* $|N| = |A| = n$. *Let* $q = (n!)^{n+|N(p)|}$, σ *be any ordering of* N^q, *and* $\mu = f^{TTC}(\sigma, \tilde{p}^q)$. *Then* $M(\mu, \tilde{p}^q) = f^{PST}(p)$.

Both ex-ante stable PS and PST are appealing generalizations of PS to $\mathscr{P}_{\text{HET}}$. The former minimally deviates from PS and satisfies a stronger envy-free notion. The latter is reduced to TTC in the special case of housing markets, and is asymptotically equivalent to the randomized TTC mechanism. Their distinction is better understood by interpreting them as random mechanisms constructed from different deterministic mechanisms using the division method. Compared with stable serial dictatorships, the TTC mechanisms are more popular due to their strategy-proofness.[44] However, neither ex-ante stable PS nor PST is strategy-proof.

7.3 A Probabilistic Version of Boston Mechanism

The *Boston mechanism* (Abdulkadiroğlu and Sönmez, 2003b) is one of the most popular school choice mechanisms in practice. Under strict priorities, it selects a deterministic allocation through a procedure similar to DA, with the only difference being that in each step the acceptance is final. This mechanism is neither stable nor strategy-proof, and its popularity mainly comes from the following welfare property, as discussed in Kojima and Ünver (2014): preference ranks are respected, in the sense that if one agent envies the object received by another agent, then the latter agent must rank the object weakly higher in her preference list.[45] This property also implies efficiency.

However, when priorities are weak and ties are broken randomly (through single or multiple tie-breaking, for example), the randomized Boston mechanism loses its attractive properties ex-ante. Harless (2018) defines the notion of **respect for rank** for a random allocation M in a problem $p = (N, A, R, \succeq)$, which requires that for any $i, j \in N$ and $a \in A$, if $F(R_i, a, M) < 1$ and $M_{ja} > 0$, then $|\{b \in A : bR_j a\}| \leq |\{b \in A : bR_i a\}|$. Respect for rank implies sd-efficiency, and he shows that the randomized Boston mechanism is not even sd-efficient. Then, instead of randomization, we can apply the Boston mechanism to continuum divided problems to generate a random mechanism similar to DC, with the key difference being that in each step the acceptance of proposals is final. Hence, in each step, only if the sum of the measures of the previously accepted

[44] It is also worth mentioning that the stable serial dictatorships can characterize the set of stable and efficient deterministic allocations for each $p \in \mathscr{P}_{\text{HET}}$.

[45] Kojima and Ünver (2014) and Doğan and Klaus (2018) use this as the central axiom to characterize the Boston mechanism in the deterministic setting.

proposals is $x < 1$, an object considers new proposals, and accepts some of them such that the sum of their measures is at most $1-x$. This probabilistic version of the Boston mechanism satisfies respect for rank, and thus sd-efficiency.[46]

7.4 Multi-Unit Demands

Finally, in house allocation problems with multi-unit demands, there are two classes of serial dictatorships. In the first class, each agent i with a demand of d_i picks the best d_i objects available when it is her turn to pick.[47] Applying such a mechanism to (finitely) divided problems can lead to the generalization of PS by Heo (2014), in which each agent's consuming rate is equal to her demand. In the second class of serial dictatorships, each agent picks only one object when it is her turn, and hence there are multiple rounds of sequential assignments.[48] Applying such a mechanism to divided problems can lead to the generalization of PS by Kojima (2009), in which each agent i with a demand of d_i consumes objects during the time interval $[0, d_i]$, at the unit rate. While the first class of serial dictatorships is strategy-proof, the second class is not. However, neither of the two generalizations of PS is strategy-proof.

8 Conclusion

We introduced an appealing fairness concept for the allocation of discrete resources under weak priorities. The main focus of our study is on an ex-ante fair mechanism and the properties of ex-ante fair allocations. The division method underlies our analysis, and also provides a new perspective on various other random mechanisms. In the end, we briefly mention two plausible directions for future research. First, it is interesting to explore the performance of DC in large markets. Given the work of Che and Kojima (2010), in fact it is reasonable to conjecture that DA with single tie-breaking and DC are asymptotically equivalent, under some regularity conditions. Second, we hope the division method can be applied to other classes of allocation or matching problems that are not necessarily priority-augmented, in order to generate new random mechanisms.

[46]In the special case of house allocation, Harless (2018) and Chen et al. (2021) propose and characterize another probabilistic version of the Boston mechanism that also satisfies respect for rank, but their mechanism is different from ours in this domain.

[47]For studies on these mechanisms, see, for example, Pápai (2000, 2001), Klaus and Miyagawa (2001) and Ehlers and Klaus (2003).

[48]See Bogolmonaia et al. (2014) for discussions on these mechanisms.

Appendix A

A.1 Proof of Theorem 1

Consider any problem $p = (N, A, R, \succeq)$. First, in the proof we will use the following crucial result regarding the deferred consumption procedure for p and the sequence of tentative assignments $\{M^k\}$, which can be easily shown using Observation 1.

Claim 1. *In any step k of the deferred consumption procedure, an agent $i \in N$ has a proposal $[x, y]$ rejected by an object $a \in A$ if and only if $F(R_i, a, M^k) = y$, $F(R_i, a, M^{k+1}) = x$, and $x < y$.*

Let $f^{\mathrm{DC}}(p) = M^*$. By construction, for all $i \in N$ and $a \in A$, $M^*_{ia} \geq 0$ and $\sum_{b \in A} M^*_{ib} \leq 1$. Suppose that some object $a \in A$ is over-assigned, i.e., $\sum_{i \in N} M^*_{ia} > 1$. Given that $\lim_{k\to\infty} \left\{ \sum_{i \in N} M^k_{ia} \right\} = \sum_{i \in N} M^*_{ia} > 1$, there exist K and $\epsilon > 0$ such that for any $k > K$, $\sum_{i \in N} M^k_{ia} > 1 + \epsilon$. This implies that for any $k > K$, the sum of the measures of the proposals rejected by a is larger than ϵ in step k of the (deferred consumption) procedure. Then, by Claim 1, $\sum_{i \in N} F(R_i, a, M^k) - \sum_{i \in N} F(R_i, a, M^{k+1}) > \epsilon$ for all $k > K$, which is clearly impossible. Therefore, M^* is a well-defined random allocation.

We next show M^* is ex-ante fair. It is individually rational as each agent never proposes to consume an object worse than her outside option. For any object $a \in A$ such that $\sum_{i \in N} M^*_{ia} < 1$, since $\lim_{k\to\infty} \left\{ \sum_{i \in N} M^k_{ia} \right\} < 1$, there exists K such that $\sum_{i \in N} M^k_{ia} < 1$ for all $k > K$. It follows that object a never rejects a proposal in the procedure. Then for any $i \in N$, given that $F(R_i, a, M^1) = 1$, Claim 1 implies that $F(R_i, a, M^k) = 1$ for all k. Hence, $F(R_i, a, M^*) = 1$, and M^* is non-wasteful.

Suppose that M^* is not ex-ante stable. Then there exist $i, j \in N$ and $a \in A$ such that $i \succ_a j$, $F(R_i, a, M^*) < 1$ and $M^*_{ja} > 0$. Since $F(R_i, a, M^1) = 1$ and $\lim_{k\to\infty} F(R_i, a, M^k) < 1$, by Claim 1, agent i has a proposal rejected by a in some step K. Then, by the choice rule of a, it does not tentatively accept any proposal from j in any step $k > K$. Since $M^*_{ja} > 0$, we have $\lim_{k\to\infty} F(P_j, a, M^k) < \lim_{k\to\infty} F(R_j, a, M^k) = F(R_j, a, M^*)$. It follows that for some step $k > K$ of the procedure, $F(P_j, a, M^k) < F(R_j, a, M^*)$. As the sequence $\left\{F(R_j, a, M^\ell)\right\}$ is decreasing, $F(P_j, a, M^k) < F(R_j, a, M^k)$. Therefore, j's proposal $\left[F(P_j, a, M^k), F(R_j, a, M^k)\right]$ is in the choice set of a in step k, and a rejects this proposal. Then, by Claim 1, $F(R_j, a, M^{k+1}) = F(P_j, a, M^k) < F(R_j, a, M^*)$, contradicting to the fact that the decreasing sequence $\left\{F(R_j, a, M^\ell)\right\}$ converges to $F(R_j, a, M^*)$.

Suppose that M^* is not ordinally fair. Then there exist $i, j \in N$ and $a \in A$ such that $i \sim_a j$, $M^*_{ia} > 0$ and $F(R_i, a, M^*) > F(R_j, a, M^*)$. Since $\lim_{k\to\infty} F(R_j, a, M^k) < F(R_i, a, M^*)$ and $F(R_j, a, M^1) = 1$, by Claim 1, in some step K, agent j has a proposal $\left[x, F(R_j, a, M^K)\right]$ rejected by a and $F(R_j, a, M^{K+1}) = x < F(R_i, a, M^*)$. By the choice rule of a, it does not tentatively accept any proposal $[y, z]$ from i such that $z > x$, in any step $k > K$. As in the above proof of ex-ante stability, $M^*_{ia} > 0$ implies that we can find some step $k > K$ of the procedure such that $F(P_i, a, M^k) < F(R_i, a, M^*) \leq F(R_i, a, M^k)$. Hence, i's proposal $\left[F(P_i, a, M^k), F(R_i, a, M^k)\right]$ is in the choice set of a in step k. Since $F(R_i, a, M^k) \geq F(R_i, a, M^*) > x$, i must have a proposal $\left[y, F(R_i, a, M^k)\right]$ rejected by a in this step, and $y \leq \max\left\{x, F(P_i, a, M^k)\right\}$. Then, by Claim 1,

$$F(R_i, a, M^{k+1}) = y \leq \max\left\{x, F(P_i, a, M^k)\right\} < F(R_i, a, M^*).$$

This leads to a contradiction as $\left\{F(R_i, a, M^\ell)\right\}$ is decreasing and converges to $F(R_i, a, M^*)$.

Finally, we show that, for any ex-ante fair allocation M for p, $M^* R^{sd}_N M$. Then it is sufficient to show that $M^k R^{sd}_N M$ for all k, and we prove this by induction. It is obvious that $M^1 R^{sd}_N M$. Suppose that for some $k \geq 1$, $M^k R^{sd}_N M$, but there exists $i \in N$ such that we do not have $M^{k+1}_i R^{sd}_i M_i$. Then for some $a \in A$, $F(R_i, a, M^{k+1}) < F(R_i, a, M) \leq F(R_i, a, M^k)$. By Claim 1, in step k, agent i's proposal $\left[F(P_i, a, M^k), F(R_i, a, M^k)\right]$ is in the choice set of object a and it rejects her proposal $\left[F(R_i, a, M^{k+1}), F(R_i, a, M^k)\right]$. Then we have

$$F(R_i, a, M) > F(R_i, a, M^{k+1}) \geq F(P_i, a, M^k) \geq F(P_i, a, M).$$

It follows that

$$M_{ia} = F(R_i, a, M) - F(P_i, a, M) > F(R_i, a, M^{k+1}) - F(P_i, a, M^k) \geq 0.$$

Consider the proposals tentatively accepted by a in step k. The sum of the measures of these proposals is 1, as i has a proposal rejected. If i has a proposal tentatively accepted by a in this step, then this proposal is $\left[F(P_i, a, M^k), F(R_i, a, M^{k+1})\right]$. The above inequality implies that the measure of this proposal is less than the probability that i receives a under M. Therefore, there must exist some $j \in N$ such that j has a proposal $\left[F(P_j, a, M^k), x\right]$ tentatively accepted by a in step k, and

$$x - F(P_j, a, M^k) > M_{ja} = F(R_j, a, M) - F(P_j, a, M). \tag{1}$$

Since i has a proposal rejected by a in this step, by the choice rule of a, $j \succeq_a i$.

If $j \succ_a i$, we have $F(R_j, a, M) = 1$, since M is ex-ante stable and $M_{ia} > 0$. Then, given $x \leq 1$, (1) implies that $F(P_j, a, M^k) < F(P_j, a, M)$, contradicting to $M^k R_N^{sd} M$. Next, consider the case that $j \sim_a i$. Since i's proposal $\left[F(R_i, a, M^{k+1}), F(R_i, a, M^k)\right]$ is rejected by a in step k, by the choice rule of a, $x \leq F(R_i, a, M^{k+1})$. Hence,

$$x \leq F(R_i, a, M^{k+1}) < F(R_i, a, M) \leq F(R_j, a, M), \tag{2}$$

where the last inequality follows from $M_{ia} > 0$ and the ordinal fairness of M. Then, (1) and (2) imply that $F(P_j, a, M^k) < F(P_j, a, M)$, and a contradiction is reached.

A.2 Proof of Lemma 1

Consider any $i \in N$ and $a \in A \cup \{i\}$. First, since every $M \in S$ is individually rational, for each $b \in A$ such that $i P_i b$, $(\vee S)_{ib} = 0$. Therefore, if $i R_i a$, then $F(R_i, a, \vee S) = (\vee S)_{ii} + \sum_{b \in A} (\vee S)_{ib} = 1 = \sup\{F(R_i, a, M) : M \in S\}$. Next, consider the case that $a P_i i$. Let $A' = \{b \in A : b R_i a\}$ and $|A'| = k$. If $k = 1$, then $F(R_i, a, \vee S) = \sup\{F(R_i, a, M) : M \in S\}$. Suppose that $k > 1$. We list the objects in A' in the order of i's preferences: let $A' = \{a_1, \ldots, a_k\}$ such that $a_1 P_i a_2, \ldots, a_{k-1} P_i a_k$, where $a_k = a$. Then,

$$\begin{aligned}
F(R_i, a_k, \vee S) &= \sum_{\ell=1}^{k} (\vee S)_{ia_\ell} \\
&= \sum_{\ell=1}^{k} \Big\{ \sup\{F(R_i, a_\ell, M) : M \in S\} - \sup\{F(P_i, a_\ell, M) : M \in S\} \Big\} \\
&= \sup\{F(R_i, a_1, M) : M \in S\} + \\
&\quad \sum_{\ell=2}^{k} \Big\{ \sup\{F(R_i, a_\ell, M) : M \in S\} - \sup\{F(R_i, a_{\ell-1}, M) : M \in S\} \Big\} \\
&= \sup\{F(R_i, a_k, M) : M \in S\}.
\end{aligned}$$

By similar arguments, it can be shown that $F(R_i, a, \wedge S) = \inf\{F(R_i, a, M) : M \in S\}$.

A.3 Proof of Theorem 2

We prove the results in Theorem 2 in a particular order. The proof consists of eight parts. We first show (iii) and (iv), in Part 1 and Part 2, by similar techniques. In Part

3, we show $\vee\{M, M'\} \in \mathscr{E}$. The first statement in the rural hospital theorem is also proved along the way. Part 4 deals with the second statement in (i). Building on these results, in Part 5, we show $\wedge\{M, M'\} \in \mathscr{E}$, and hence $(\mathscr{E}, R_N^{sd})$ is a lattice. Part 6 and Part 7 establish that this lattice is complete. Finally, we show the second statement in the rural hospital theorem in Part 8.

Part 1: two-sided efficiency.

Assume to the contrary, there exists some allocation $M'' \neq M$ such that $M'' R_N^{sd} M$ and $M'' \succeq_A^{sd} M$. Let $N' = \{i \in N : M_i \neq M_i''\}$. Clearly $N' \neq \emptyset$. For each $i \in N'$, since $M_i'' R_i^{sd} M_i$ and M is individually rational, $\{a \in A : aP_i i, M_{ia}'' > M_{ia}\} \neq \emptyset$. Define a_i as the maximal element in this set under R_i. Then $M_{ia}'' = M_{ia}$ for all $a \in A$ such that $aP_i a_i$.

Now, consider any $i \in N'$. Since $F(R_i, a_i, M) < F(R_i, a_i, M'') \leq 1$ and M is ex-ante stable, we have $F(\succeq_{a_i}, i, M) = 1$. Then $M_{a_i}'' \succeq_{a_i}^{sd} M_{a_i}$ implies $F(\succeq_{a_i}, i, M'') = 1$ and $F(\succ_{a_i}, i, M'') \geq F(\succ_{a_i}, i, M)$. Hence $\sum_{j \in N : j \sim_{a_i} i} M_{ja_i}'' \leq \sum_{j \in N : j \sim_{a_i} i} M_{ja_i}$. Since $M_{ia_i}'' > M_{ia_i}$, there exists some $j \in N \setminus \{i\}$ such that $j \sim_{a_i} i$ and $M_{ja_i}'' < M_{ja_i}$. Then $j \in N'$ and $a_j P_j a_i$. As M is ordinally fair and $M_{ja_i} > 0$, we have

$$F(R_i, a_i, M) \geq F(R_j, a_i, M) > F(R_j, a_j, M).$$

In sum, it has been shown that for any $i \in N'$, there exists $j \in N'$ such that $F(R_i, a_i, M) > F(R_j, a_j, M)$. This leads to a contradiction as N' is finite.

Part 2: conflicting interests.

Suppose that $MR_N^{sd} M'$, but for some $a \in A$ and $i \in N$, $F(\succeq_a, i, M) > F(\succeq_a, i, M')$. Then there exists $j \in N$ such that $j \succeq_a i$ and $M_{ja} > M_{ja}'$. Since $F(\succeq_a, j, M') \leq F(\succeq_a, i, M') < 1$ and M' is ex-ante stable, $F(R_j, a, M') = 1$. Then

$$F(P_j, a, M') = 1 - M_{ja}' > 1 - M_{ja} \geq 1 - \sum_{b \in A \cup \{j\} : aR_j b} M_{jb} = F(P_j, a, M).$$

This contradicts to $M_j R_j^{sd} M_j'$.

To show the other direction, suppose that $M' \succeq_A^{sd} M$, but we do not have $MR_N^{sd} M'$. Let $N' = \{i \in N : F(R_i, a, M') > F(R_i, a, M)$ for some $a \in A$ such that $aP_i i\}$. Then, given that M is individually rational, $N' \neq \emptyset$. For each $i \in N'$, define a_i as the maximal object in the set $\{a \in A : aP_i i, F(R_i, a, M') > F(R_i, a, M)\}$ under R_i.

Consider any $i \in N'$. By the definition of a_i, we have $M'_{ia_i} > M_{ia_i}$. Since $F(R_i, a_i, M) < 1$ and M is ex-ante stable, $F(\succeq_{a_i}, i, M) = 1$. It follows from $M' \succeq^{sd}_A M$ that $F(\succeq_{a_i}, i, M') = 1$ and $\sum_{j \in N: j \sim_{a_i} i} M'_{ja_i} \leq \sum_{j \in N: j \sim_{a_i} i} M_{ja_i}$. Therefore, $M'_{ia_i} > M_{ia_i}$ implies the existence of some $j \in N \setminus \{i\}$ such that $j \sim_{a_i} i$ and $M'_{ja_i} < M_{ja_i}$. Since M' is ordinally fair and $M'_{ia_i} > 0$,

$$F(R_j, a_i, M') \geq F(R_i, a_i, M') > F(R_i, a_i, M). \tag{3}$$

Since M is ordinally fair and $M_{ja_i} > 0$,

$$F(R_i, a_i, M) \geq F(R_j, a_i, M). \tag{4}$$

Hence (3) and (4) imply $F(R_j, a_i, M') > F(R_j, a_i, M)$. So $j \in N'$. Since $M'_{ja_i} < M_{ja_i}$, $a_j P_j a_i$. Then by (4) and $M_{ja_i} > 0$,

$$F(R_i, a_i, M) \geq F(R_j, a_i, M) > F(R_j, a_j, M).$$

In sum, we have shown that for any $i \in N'$, there exists $j \in N'$ such that $F(R_i, a_i, M) > F(R_j, a_j, M)$, which leads to a contradiction since N' is finite.

Part 3: $\vee\{M, M'\} \in \mathscr{E}$, and the first part of rural hospital theorem.

Denote $\bar{M} = \vee\{M, M'\}$. By the construction, $\bar{M}_{ia} \geq 0$ for all $i \in N$ and $a \in A$. Then by Lemma 1, for any $i \in N$, $\sum_{a \in A} \bar{M}_{ia} = F(P_i, i, \bar{M}) = \max\{F(P_i, i, M), F(P_i, i, M')\} \leq 1$. Therefore, for $\bar{M}$ to be a well-defined random allocation, it remains to show that the probability shares of any object are not over-assigned.

Lemma 2. *For any $M^1, M^2 \in \mathscr{E}$, $i \in N$ and $a \in A$, if $F(R_i, a, M^1) \geq F(R_i, a, M^2)$, then $M^1_{ia} \geq \left(\vee\{M^1, M^2\}\right)_{ia}$.*

Proof of Lemma 2.

$$\begin{aligned}
M^1_{ia} &= F(R_i, a, M^1) - F(P_i, a, M^1) \\
&= \max\left\{F(R_i, a, M^1), F(R_i, a, M^2)\right\} - F(P_i, a, M^1) \\
&\geq \max\left\{F(R_i, a, M^1), F(R_i, a, M^2)\right\} - \max\left\{F(P_i, a, M^1), F(P_i, a, M^2)\right\} \\
&= \left(\vee\{M^1, M^2\}\right)_{ia}
\end{aligned}$$

■

Claim 2. *For any $a \in A$, either $\bar{M}_{ia} \leq M_{ia}$ for all $i \in N$, or $\bar{M}_{ia} \leq M'_{ia}$ for all $i \in N$.*

Proof of Claim 2. Consider any $a \in A$. Let $N(a) = \{i \in N : \bar{M}_{ia} > 0\}$. If $F(R_i, a, M) = F(R_i, a, M')$ for all $i \in N(a)$, then Claim 2 follows directly from Lemma 2. Now, suppose that there exists $i \in N(a)$ such that $F(R_i, a, M) \neq F(R_i, a, M')$. Without loss of generality, let $F(R_i, a, M) > F(R_i, a, M')$. Then by Lemma 2, $M_{ia} \geq \bar{M}_{ia} > 0$. Next, we want to show that $F(R_j, a, M) \geq F(R_j, a, M')$ for all $j \in N(a) \setminus \{i\}$. Assume to the contrary, for some $j \in N(a)$, $F(R_j, a, M) < F(R_j, a, M')$. Then by Lemma 2, $M'_{ja} \geq \bar{M}_{ja} > 0$. Since M is ex-ante stable, $M_{ia} > 0$ and $F(R_j, a, M) < F(R_j, a, M') \leq 1$ imply $i \succeq_a j$. Similarly, given that M' is ex-ante stable, $M'_{ja} > 0$ and $F(R_i, a, M') < F(R_i, a, M) \leq 1$ imply $j \succeq_a i$. Hence, $i \sim_a j$. Then we have

$$F(R_i, a, M') \geq F(R_j, a, M') > F(R_j, a, M) \geq F(R_i, a, M),$$

where the first inequality follows from $M'_{ja} > 0$ and the ordinal fairness of M', and the last inequality follows from $M_{ia} > 0$ and the ordinal fairness of M. This contradicts to the assumption that $F(R_i, a, M) > F(R_i, a, M')$. Therefore, $F(R_k, a, M) \geq F(R_k, a, M')$ for all $k \in N(a)$. By Lemma 2, $\bar{M}_{ka} \leq M_{ka}$ for all $k \in N$. ■

It follows immediately from Claim 2 that for every $a \in A$, $\sum_{i \in N} \bar{M}_{ia} \leq 1$. So $\bar{M}$ is a well-defined random allocation. Next, we show that $\bar{M}$ is ex-ante fair. Lemma 1 implies that $\bar{M}$ is individually rational. The non-wastefulness of $\bar{M}$ can be deduced from the following critical result by Erdil (2014).

Lemma 3 (Reshuffling Lemma)**.** *Let M^1 and M^2 be two random allocations. If M^1 is individually rational, non-wasteful and $M^2 R_N^{sd} M^1$, then $\sum_{b \in A} M^1_{ib} = \sum_{b \in A} M^2_{ib}$ for every $i \in N$, and $\sum_{j \in N} M^1_{ja} = \sum_{j \in N} M^2_{ja}$ for every $a \in A$.*

If $\bar{M}$ is wasteful, then there exist $i \in N$ and $a \in A$ such that $F(R_i, a, \bar{M}) < 1$ and $\sum_{j \in N} \bar{M}_{ja} < 1$. As $\bar{M} R_N^{sd} M$, $F(R_i, a, M) < 1$. Then, since M is non-wasteful, $\sum_{j \in N} M_{ja} = 1$, contradicting to Lemma 3.

We pause the proof of the ex-ante fairness of $\bar{M}$ and present two other important implications of the Reshuffling Lemma.

- First, since $\bar{M} R_N^{sd} M$, $\bar{M} R_N^{sd} M'$, and both M and M' are individually rational and non-wasteful, Lemma 3 and Claim 2 together imply the following result.

  Claim 3. *For any $a \in A$, either $\bar{M}_a = M_a$ or $\bar{M}_a = M'_a$.*

- Second, for any $i \in N$ and $a \in A$, by Lemma 3 we have

$$\sum_{b \in A} M_{ib} = \sum_{b \in A} \bar{M}_{ib} = \sum_{b \in A} M'_{ib},$$

$$\sum_{j \in N} M_{ja} = \sum_{j \in N} \bar{M}_{ja} = \sum_{j \in N} M'_{ja}.$$

  That is, we have shown the first part of the rural hospital theorem.

Now, suppose that $\bar{M}$ is not ex-ante stable, then there exist $i, j \in N$ and $a \in A$ such that $i \succ_a j$, $\bar{M}_{ja} > 0$ and $F(R_i, a, \bar{M}) < 1$. Clearly, $M''_{ja} > 0$ for some $M'' \in \{M, M'\}$. But $\bar{M} R^{sd}_N M''$ implies $F(R_i, a, M'') \leq F(R_i, a, \bar{M}) < 1$, contradicting to the ex-ante stability of M''. Finally, suppose that $\bar{M}$ is not ordinally fair, then there exist $i, j \in N$ and $a \in A$ such that $i \sim_a j$, $\bar{M}_{ia} > 0$ and $F(R_i, a, \bar{M}) > F(R_j, a, \bar{M})$. By Lemma 1, $F(R_i, a, \bar{M}) = \max\{F(R_i, a, M), F(R_i, a, M')\}$. Without loss of generality, let $F(R_i, a, \bar{M}) = F(R_i, a, M) \geq F(R_i, a, M')$. Then by Lemma 2, $M_{ia} \geq \bar{M}_{ia} > 0$. But we have

$$F(R_j, a, M) \leq F(R_j, a, \bar{M}) < F(R_i, a, \bar{M}) = F(R_i, a, M),$$

contradicting to the ordinal fairness of M. In sum, $\bar{M}$ is ex-ante fair.

Part 4: for any $i \in N$, either $M_i R^{sd}_i M'_i$ or $M'_i R^{sd}_i M_i$; for any $a \in A$, either $M_a \succeq^{sd}_a M'_a$ or $M'_a \succeq^{sd}_a M_a$.

First, consider any $a \in A$. By Claim 3, $\bar{M}_a = M_a$ or $\bar{M}_a = M'_a$. Since $\bar{M}$ is ex-ante fair, $\bar{M} R^{sd}_N M$, and $\bar{M} R^{sd}_N M'$, Part 2 (conflicting interests) implies that $M \succeq^{sd}_A \bar{M}$ and $M' \succeq^{sd}_A \bar{M}$. Therefore, $M'_a \succeq^{sd}_a M_a$ if $\bar{M}_a = M_a$, and $M_a \succeq^{sd}_a M'_a$ if $\bar{M}_a = M'_a$.

Second, assume to the contrary, for some $i \in N$, M_i and M'_i are not comparable using the first-order stochastic dominance relation. Since M and M' are individually rational, there exist $a, b \in A$ such that $a P_i i$, $b P_i i$, $F(R_i, a, M) > F(R_i, a, M')$ and $F(R_i, b, M) < F(R_i, b, M')$. Without loss of generality, assume $a P_i b$. Let c be the worst object in $\{a' \in A : a' P_i b, F(R_i, a', M) > F(R_i, a', M')\}$, and d the object next to c (and worse than c) on i's preference list. Then we have $a R_i c P_i d R_i b$, $F(R_i, c, M) > F(R_i, c, M')$

and $F(R_i, d, M) \le F(R_i, d, M')$. Moreover,

$$\begin{aligned} M'_{id} &= F(R_i, d, M') - F(R_i, c, M') \\ &> F(R_i, d, M') - F(R_i, c, M) \\ &= \max\left\{F(R_i, d, M'), F(R_i, d, M)\right\} - \max\left\{F(R_i, c, M'), F(R_i, c, M)\right\} \\ &= \bar{M}_{id}. \end{aligned}$$

By Claim 3, $M'_{id} > \bar{M}_{id}$ implies $\bar{M}_d = M_d$. It follows from Part 2 (conflicting interests) that $M'_d \succeq_d^{sd} \bar{M}_d = M_d$. Since $F(R_i, d, M) \le F(R_i, b, M) < F(R_i, b, M') \le 1$ and M is ex-ante stable, $F(\succeq_d, i, M) = 1$. Therefore, $F(\succeq_d, i, M') = 1$ and $\sum_{j \in N : j \sim_d i} M'_{jd} \le \sum_{j \in N : j \sim_d i} M_{jd}$. Then $M'_{id} > \bar{M}_{id} = M_{id}$ implies that there exists $j \in N$ such that $j \sim_d i$ and $M'_{jd} < M_{jd}$, and we have

$$F(R_j, d, M') \ge F(R_i, d, M') \ge F(R_i, d, M) \ge F(R_j, d, M),$$

where the first inequality follows from $M'_{id} > 0$ and the ordinal fairness of M', and the last inequality follows from $M_{jd} > 0$ and the ordinal fairness of M. Since $F(R_j, d, M') \ge F(R_j, d, M)$, by Lemma 2, $\bar{M}_{jd} \le M'_{jd} < M_{jd}$. This is a contradiction since $\bar{M}_d = M_d$.

Part 5: $\wedge\{M, M'\} \in \mathscr{E}$.

Denote $\hat{M} = \wedge\{M, M'\}$. Let $N' = \{i \in N : M'_i R_i^{sd} M_i\}$. If $i \in N'$, then $\bar{M}_i = M'_i$ and $\hat{M}_i = M_i$. By Part 4, if $i \in N \setminus N'$, then $\bar{M}_i = M_i$ and $\hat{M}_i = M'_i$. It follows that $\hat{M}_{ia} \ge 0$ and $\sum_{b \in A} \hat{M}_{ib} \le 1$ for all $i \in N$ and $a \in A$. For any $a \in A$, by the first part of the rural hospital theorem,

$$\sum_{i \in N'} M_{ia} + \sum_{i \in N \setminus N'} M_{ia} = \sum_{i \in N} M_{ia} = \sum_{i \in N} \bar{M}_{ia} = \sum_{i \in N'} M'_{ia} + \sum_{i \in N \setminus N'} M_{ia}.$$

Then

$$\sum_{i \in N'} M_{ia} = \sum_{i \in N'} M'_{ia}.$$

This implies that

$$\sum_{i \in N} \hat{M}_{ia} = \sum_{i \in N'} M_{ia} + \sum_{i \in N \setminus N'} M'_{ia} = \sum_{i \in N'} M'_{ia} + \sum_{i \in N \setminus N'} M'_{ia} = \sum_{i \in N} M'_{ia} \le 1.$$

That is, the probability shares of a are not over-assigned. So $\hat{M}$ is a well-defined random allocation. We show that it is ex-ante fair. First, $\hat{M}$ is individually rational by Lemma 1. Second, suppose that for some $a \in A$, $\sum_{i\in N} \hat{M}_{ia} < 1$. It was shown above that $\sum_{i\in N} M'_{ia} = \sum_{i\in N} \hat{M}_{ia}$. The first part of the rural hospital theorem further implies that $\sum_{i\in N} M_{ia} = \sum_{i\in N} M'_{ia} = \sum_{i\in N} \hat{M}_{ia} < 1$. Since M and M' are non-wasteful, for all $i \in N$, we have $F(R_i, a, M) = F(R_i, a, M') = 1$. It follows that $F(R_i, a, \hat{M}) = 1$ for all $i \in N$, and hence $\hat{M}$ is non-wasteful.

Suppose that $\hat{M}$ is not ex-ante stable. Then there exist $i, j \in N$ and $a \in A$ such that $i \succ_a j$, $F(R_i, a, \hat{M}) < 1$ and $\hat{M}_{ja} > 0$. Without loss of generality, assume $\hat{M}_i = M_i$. Since M is ex-ante stable, $F(\succ_a, j, M) = 1$. In particular, $M_{ja} = 0$. Then $M'_{ja} = \hat{M}_{ja} > 0$ and $M_j R_j^{sd} M'_j$. It follows that $\bar{M}_{ja} = M_{ja} \neq M'_{ja}$. By Claim 3, $\bar{M}_a = M_a$. Then, as $\bar{M} R_N^{sd} M'$, Part 2 (conflicting interests) implies $M'_a \succeq_a^{sd} M_a$. However, this contradicts to the facts that $M'_{ja} > 0$ and $F(\succ_a, j, M) = 1$.

Finally, suppose that $\hat{M}$ is not ordinally fair. Then there exist $i, j \in N$ and $a \in A$ such that $i \sim_a j$, $\hat{M}_{ia} > 0$ and $F(R_i, a, \hat{M}) > F(R_j, a, \hat{M})$. Without loss of generality, let $\hat{M}_i = M_i$. Then by the ordinal fairness of M, $\hat{M}_j = M'_j$. Since $M'_i R_i^{sd} M_i$,

$$F(R_i, a, M') \geq F(R_i, a, M) = F(R_i, a, \hat{M}) > F(R_j, a, \hat{M}) = F(R_j, a, M').$$

By the ordinal fairness of M', $M'_{ia} = 0$. Given that $M'_{ia} \neq M_{ia}$ and $\bar{M}_{ia} = M'_{ia}$, by Claim 3 we have $\bar{M}_a = M'_a$. Then Part 2 (conflicting interests) implies $M_a \succeq_a^{sd} M'_a$.

Since $F(R_j, a, M') < F(R_i, a, M) \leq 1$, the ex-ante stability of M' implies $F(\succeq_a, j, M') = 1$. Then $F(\succeq_a, j, M) = 1$ and $\sum_{k\in N: k\sim_a j} M_{ka} \leq \sum_{k\in N: k\sim_a j} M'_{ka}$. As $M_{ia} > M'_{ia}$ and $i \sim_a j$, there exists $k \in N$ such that $k \sim_a j$ and $M_{ka} < M'_{ka}$. Since $\bar{M}_a = M'_a$, we have $\bar{M}_k = M'_k$. Hence $M'_k R_k^{sd} M_k$. Then

$$F(R_k, a, M') \geq F(R_k, a, M) \geq F(R_i, a, M) > F(R_j, a, M'),$$

where the second inequality follows from the ordinal fairness of M and $M_{ia} > 0$. However, given that $M'_{ka} > 0$, this contradicts to the ordinal fairness of M'.

Part 6: $\vee S \in \mathscr{E}$.

First, given that any agent can compare the lotteries obtained under two ex-ante fair allocations using the first-order stochastic dominance relation (Part 4), it is straightforward to see that for any $M'' \in \mathscr{E}$, and any non-empty and finite set $S' \subseteq \mathscr{E}$ such that $\vee S' \in \mathscr{E}$, we have $\vee \{S' \cup \{M''\}\} = \vee \{\vee S', M''\}$. Therefore, by Part 3 and an induction argument, for any non-empty and finite $S'' \subseteq \mathscr{E}$, $\vee S'' \in \mathscr{E}$.

Now, consider the set $S \subseteq \mathscr{E}$, which can be infinite. As in the case of $\bar{M}$, it can be easily shown that for all $i \in N$ and $a \in A$, $(\vee S)_{ia} \geq 0$ and $\sum_{b \in A}(\vee S)_{ib} \leq 1$. For $\vee S$ to be a well-defined random allocation, it remains to show that $\sum_{i \in N}(\vee S)_{ia} \leq 1$ for every $a \in A$. Assume to the contrary, $\sum_{i \in N}(\vee S)_{ia} > 1$ for some $a \in A$. For each $i \in N$, we can find $M(i) \in S$ such that

$$F(R_i, a, M(i)) > \sup\left\{F(R_i, a, M'') : M'' \in S\right\} - \frac{1}{|N|}\left\{\sum_{j \in N}(\vee S)_{ja} - 1\right\}.$$

Let $S' = \{M(i) : i \in N\} \subseteq S$. Then for each $i \in N$,

$$\begin{aligned}
(\vee S')_{ia} &= \max\left\{F(R_i, a, M'') : M'' \in S'\right\} - \max\left\{F(P_i, a, M'') : M'' \in S'\right\} \\
&\geq F(R_i, a, M(i)) - \sup\left\{F(P_i, a, M'') : M'' \in S\right\} \\
&> \sup\left\{F(R_i, a, M'') : M'' \in S\right\} - \frac{1}{|N|}\left\{\sum_{j \in N}(\vee S)_{ja} - 1\right\} - \sup\left\{F(P_i, a, M'') : M'' \in S\right\} \\
&= (\vee S)_{ia} - \frac{1}{|N|}\left\{\sum_{j \in N}(\vee S)_{ja} - 1\right\}.
\end{aligned}$$

Summing over N, we have

$$\sum_{i \in N}(\vee S')_{ia} > 1.$$

A contradiction is reached, since S' is finite and $\vee S' \in \mathscr{E}$.

Next, we show that the allocation $\vee S$ is ex-ante fair. First, the individual rationality of $\vee S$ follows from Lemma 1. As in the case of $\bar{M}$, the non-wastefulness of $\vee S$ can be deduced from the Reshuffling Lemma, since $(\vee S) R_N^{sd} M$, and M is individually rational and non-wasteful. Second, suppose that $\vee S$ is not ex-ante stable, then there exist $i, j \in N$ and $a \in A$ such that $i \succ_a j, (\vee S)_{ja} > 0$ and $F(R_i, a, \vee S) < 1$. Clearly $M''_{ja} > 0$ for some $M'' \in S$. But $F(R_i, a, M'') \leq F(R_i, a, \vee S) < 1$, contradicting to the ex-ante stability of M''. Finally, suppose that $\vee S$ is not ordinally fair. Then there exist $i, j \in N$ and $a \in A$ such that $i \sim_a j, (\vee S)_{ia} > 0$, and $F(R_i, a, \vee S) > F(R_j, a, \vee S)$. Pick some number x such that

$$\max\left\{F(R_j, a, \vee S), F(P_i, a, \vee S)\right\} < x < F(R_i, a, \vee S).$$

Since $F(R_i, a, \vee S) = \sup\{F(R_i, a, M'') : M'' \in S\}$, there exists $M'' \in S$ such that

$F(R_i, a, M'') > x$. Then $F(R_i, a, M'') > F(R_j, a, \vee S) \geq F(R_j, a, M'')$. However, $M''_{ia} = F(R_i, a, M'') - F(P_i, a, M'') > x - F(P_i, a, \vee S) > 0$, contradicting to the ordinal fairness of M''.

Part 7: $\wedge S \in \mathscr{E}$.

First, as in Part 6, it is easy to show that for any non-empty and finite $S' \subseteq \mathscr{E}$, $\wedge S' \in \mathscr{E}$. It is also straightforward to see that for all $i \in N$ and $a \in A$, $(\wedge S)_{ia} \geq 0$ and $\sum_{b \in A} (\wedge S)_{ib} \leq 1$. Below, we show that the probability shares of any object are not over-assigned in $\wedge S$, using arguments similar to those in Part 6. Assume to the contrary, for some $a \in A$, $\sum_{i \in N} (\wedge S)_{ia} > 1$. For each $i \in N$, there exists $M^1(i) \in S$ such that

$$F(P_i, a, M^1(i)) < \inf \left\{ F(P_i, a, M'') : M'' \in S \right\} + \frac{1}{|N|} \left\{ \sum_{j \in N} (\wedge S)_{ja} - 1 \right\}.$$

Define $S^1 = \left\{ M^1(i) : i \in N \right\} \subseteq S$. Then for each $i \in N$,

$$\begin{aligned}
(\wedge S^1)_{ia} &= \min \left\{ F(R_i, a, M'') : M'' \in S^1 \right\} - \min \left\{ F(P_i, a, M'') : M'' \in S^1 \right\} \\
&\geq \inf \left\{ F(R_i, a, M'') : M'' \in S \right\} - F(P_i, a, M^1(i)) \\
&> \inf \left\{ F(R_i, a, M'') : M'' \in S \right\} - \left(\inf \left\{ F(P_i, a, M'') : M'' \in S \right\} + \frac{1}{|N|} \left\{ \sum_{j \in N} (\wedge S)_{ja} - 1 \right\} \right) \\
&= (\wedge S)_{ia} - \frac{1}{|N|} \left\{ \sum_{j \in N} (\wedge S)_{ja} - 1 \right\}.
\end{aligned}$$

Summing over N, we have

$$\sum_{i \in N} (\wedge S^1)_{ia} > 1.$$

This contradicts to the fact that S^1 is finite and $\wedge S^1 \in \mathscr{E}$. Hence, $\wedge S$ is a well-defined random allocation. We show that it is ex-ante fair. First, the individual rationality follows from Lemma 1. Second, to see non-wastefulness, consider any $a \in A$ such that $\sum_{i \in N} (\wedge S)_{ia} < 1$. For each $i \in N$, there exists $M^2(i) \in S$ such that

$$F(R_i, a, M^2(i)) < \inf \left\{ F(R_i, a, M'') : M'' \in S \right\} + \frac{1}{|N|} \left\{ 1 - \sum_{j \in N} (\wedge S)_{ja} \right\}.$$

Let $S^2 = \left\{ M^2(i) : i \in N \right\} \subseteq S$. By similar arguments as above, it can be shown that

$\sum_{i\in N}(\wedge S^2)_{ia} < 1$. Since S^2 is finite, $\wedge S^2 \in \mathscr{E}$. By the first part of the rural hospital theorem, for every $M'' \in \mathscr{E}$, $\sum_{i\in N} M''_{ia} = \sum_{i\in N}(\wedge S^2)_{ia} < 1$. Then for every $M'' \in \mathscr{E}$, by non-wastefulness, $F(R_i, a, M'') = 1$ for all $i \in N$. It follows that $F(R_i, a, \wedge S) = 1$ for all $i \in N$, and hence $\wedge S$ is non-wasteful.

Third, suppose that $\wedge S$ is not ex-ante stable. Then there exist $i, j \in N$ and $a \in A$ such that $i \succ_a j, (\wedge S)_{ja} > 0$ and $F(R_i, a, \wedge S) < 1$. Given that

$$\inf\{F(R_i, a, M'') : M'' \in S\} < 1, \text{ and}$$

$$\inf\{F(P_j, a, M'') : M'' \in S\} < \inf\{F(R_j, a, M'') : M'' \in S\},$$

we can find some $M^1, M^2 \in S$ such that

$$F(R_i, a, M^1) < 1, \text{ and } F(P_j, a, M^2) < \inf\{F(R_j, a, M'') : M'' \in S\}.$$

This implies that $F(R_i, a, \wedge\{M^1, M^2\}) < 1$ and $(\wedge\{M^1, M^2\})_{ja} > 0$, contradicting to the ex-ante stability of $\wedge\{M^1, M^2\}$.

Finally, suppose that $\wedge S$ is not ordinally fair. Then there exist $i, j \in N$ and $a \in A$ such that $i \sim_a j, (\wedge S)_{ia} > 0$, and $F(R_i, a, \wedge S) > F(R_j, a, \wedge S)$. Since

$$\inf\{F(P_i, a, M'') : M'' \in S\} < \inf\{F(R_i, a, M'') : M'' \in S\}, \text{ and}$$

$$\inf\{F(R_j, a, M'') : M'' \in S\} < \inf\{F(R_i, a, M'') : M'' \in S\},$$

we can find $M^3, M^4 \in S$ such that

$$F(P_i, a, M^3) < \inf\{F(R_i, a, M'') : M'' \in S\}, \text{ and}$$

$$F(R_j, a, M^4) < \inf\{F(R_i, a, M'') : M'' \in S\}.$$

It follows that $(\wedge\{M^3, M^4\})_{ia} > 0$ and $F(R_j, a, \wedge\{M^3, M^4\}) < F(R_i, a, \wedge\{M^3, M^4\})$, contradicting to the ordinal fairness of $\wedge\{M^3, M^4\}$.

Part 8: the second part of rural hospital theorem.

We want to show that for all $i \in N$ and $a \in A$, $M_i = M'_i$ if $\sum_{b\in A} M_{ib} < 1$, and $M_a = M'_a$ if $\sum_{j\in N} M_{ja} < 1$.

First, consider any $a \in A$ such that $\sum_{i\in N} M_{ia} < 1$. By Part 6, $\vee\mathscr{E} \in \mathscr{E}$. To prove that $M_a = M'_a$, it is sufficient to show that for any $M'' \in \mathscr{E}$, $M''_a = (\vee\mathscr{E})_a$. Assume to the contrary, $M'' \in \mathscr{E}$ and $M''_a \neq (\vee\mathscr{E})_a$. By the first part of the rural hospital theorem, $\sum_{i\in N} M''_{ia} = \sum_{i\in N}(\vee\mathscr{E})_{ia} < 1$. So there exists $i \in N$ such that $M''_{ia} < (\vee\mathscr{E})_{ia}$. Moreover, as M'' and $\vee\mathscr{E}$ are non-wasteful, $F(R_i, a, M'') = F(R_i, a, \vee\mathscr{E}) = 1$. Therefore, $F(P_i, a, M'') = 1 - M''_{ia} > 1 - (\vee\mathscr{E})_{ia} = F(P_i, a, \vee\mathscr{E})$, contradicting to the fact that $(\vee\mathscr{E})_i R^{sd}_i M''_i$.

Second, consider any $i \in N$ such that $\sum_{a\in A} M_{ia} < 1$. We show that for any $M'' \in \mathscr{E}$, $M''_i = (\wedge\mathscr{E})_i$, where $\wedge\mathscr{E} \in \mathscr{E}$ by Part 7. Suppose that $M'' \in \mathscr{E}$ and $M''_i \neq (\wedge\mathscr{E})_i$. Let a be the worst object in the set $\{b \in A : M''_{ib} \neq (\wedge\mathscr{E})_{ib}\}$, according to R_i. Clearly, $aP_i i$. By the first part of the rural hospital theorem, $\sum_{b\in A} M''_{ib} = \sum_{b\in A}(\wedge\mathscr{E})_{ib} < 1$. It follows that $F(R_i, a, M'') = F(R_i, a, \wedge\mathscr{E})$, since $M''_{ib} = (\wedge\mathscr{E})_{ib}$ for all $b \in A$ such that $aP_i b$. Then $M''_i R^{sd}_i (\wedge\mathscr{E})_i$ implies that $F(P_i, a, M'') \geq F(P_i, a, \wedge\mathscr{E})$, and hence $M''_{ia} < (\wedge\mathscr{E})_{ia}$.

Since $F(R_i, a, M'') = F(R_i, a, \wedge\mathscr{E}) \leq \sum_{b\in A}(\wedge\mathscr{E})_{ib} < 1$, by ex-ante stability we have $F(\succeq_a, i, M'') = F(\succeq_a, i, \wedge\mathscr{E}) = 1$. By Part 2 (conflicting interests), $(\wedge\mathscr{E})_a \succeq^{sd}_a M''_a$. Then $M''_{ia} < (\wedge\mathscr{E})_{ia}$ implies that there exists some $j \in N$ such that $i \sim_a j$ and $M''_{ja} > (\wedge\mathscr{E})_{ja}$. Since $M''_j R^{sd}_j (\wedge\mathscr{E})_j$, $F(P_j, a, M'') \geq F(P_j, a, \wedge\mathscr{E})$. Thus, $F(R_j, a, M'') > F(R_j, a, \wedge\mathscr{E})$, and we have

$$F(R_i, a, M'') \geq F(R_j, a, M'') > F(R_j, a, \wedge\mathscr{E}) \geq F(R_i, a, \wedge\mathscr{E}),$$

where the first inequality follows from the ordinal fairness of M'' and $M''_{ja} > 0$, and the last inequality follows from the ordinal fairness of $\wedge\mathscr{E}$ and $(\wedge\mathscr{E})_{ia} > 0$. This is a contradiction since it was already shown that $F(R_i, a, M'') = F(R_i, a, \wedge\mathscr{E})$.

A.4 Proof of Proposition 1

Let $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HA}}$ with $|N| = |A| = n$. Consider the allocation procedure in the definition of EPS, which terminates in some step $\bar{k}$. By construction, we have $\bar{k} \leq n$, and $\{\lambda_k\}_{k=1}^{\bar{k}}$ is a sequence of positive numbers with $\sum_{k=1}^{\bar{k}} \lambda_k = 1$. For each $k \in \{1, \ldots, \bar{k}\}$ and $i \in N_k$, define $k(i) \in \{0, \ldots, k-1\}$ such that $i \in N_{k(i)}$, and $i \notin N_\ell$ if $k(i) < \ell < k$, where we set $N_0 = N$. Then $d_{k-1}(i) = \lambda_{k(i)+1} + \ldots + \lambda_{k-1}$ if $d_{k-1}(i) > 0$.

Let $q = (n!)^n$. We first use induction to show that $\lambda_k q$ is an integer for every $k \in \{1, \ldots, \bar{k}\}$. Since $\lambda_1 = \frac{|E_1|}{|N_1|}$, $\lambda_1 \cdot n!$ is an integer. Assume that $\lambda_{k-1} \cdot (n!)^{k-1}$ is an integer, where $1 < k \leq \bar{k}$. Then

$$\begin{aligned}\lambda_k \cdot (n!)^k &= \frac{|E_k| - \sum_{i \in N_k} d_{k-1}(i)}{|N_k|} \cdot (n!)^k \\ &= \frac{n!}{|N_k|} \cdot \left\{ |E_k| \cdot (n!)^{k-1} - \sum_{i \in N_k} d_{k-1}(i) \cdot (n!)^{k-1} \right\},\end{aligned}$$

which is an integer since for each $i \in N_k$ with $d_{k-1}(i) > 0$,

$$d_{k-1}(i) \cdot (n!)^{k-1} = \lambda_{k(i)+1} \cdot (n!)^{k-1} + \ldots + \lambda_{k-1} \cdot (n!)^{k-1}$$

is an integer. Given that $\bar{k} \leq n$ and $q = (n!)^n$, $\lambda_k q$ is an integer for all $k \in \{1, \ldots, \bar{k}\}$.

Next, we construct a deterministic allocation for p^q based on EPS. Consider any $k \in \{1, \ldots, \bar{k}\}$. Define

$$E_k^q = \{a^x \in A^q : a \in E_k, x = 1, \ldots, q\},$$

and

$$N_k^q = \left\{ i^x \in N^q : i \in N_k, \sum_{\ell=0}^{k(i)} \lambda_\ell q + 1 \leq x \leq \sum_{\ell=0}^{k} \lambda_\ell q \right\},$$

where $\lambda_0 = 0$. Note that, since every $\lambda_\ell q$ is an integer, for each $i \in N_k$ we have

$$\left| \{i^x : i^x \in N_k^q\} \right| = \sum_{\ell=k(i)+1}^{k} \lambda_\ell q = (d_{k-1}(i) + \lambda_k) q. \tag{5}$$

We argue that the objects E_k^q can be assigned to the agents N_k^q such that each $i^x \in N_k^q$ receives some $a^y \in E_k^q$ with $a \in B_i(A_k)$. If this is not true, then by Hall's theorem, there exists $\tilde{N}_k^q \subseteq N_k^q$ such that $|\tilde{N}_k^q| > |B_{N'}(A_k)| q$, where $N' = \{i \in N_k : i^x \in \tilde{N}_k^q \text{ for some } x\}$. It follows that

$$\left| \{i^x : i \in N', i^x \in N_k^q\} \right| \geq |\tilde{N}_k^q| > \left| B_{N'}(A_k) \right| q.$$

Then by Equation 5,

$$\begin{aligned}&\sum_{i \in N'} (d_{k-1}(i) + \lambda_k) q > \left| B_{N'}(A_k) \right| q \\ \Longrightarrow\ &\lambda_k > \frac{\left| B_{N'}(A_k) \right| - \sum_{i \in N'} d_{k-1}(i)}{|N'|},\end{aligned}$$

which contradicts to the definition of λ_k.

Therefore, we can construct a deterministic allocation φ such that for each $k \in \{1, \dots, \bar{k}\}$ and $i^x \in N_k^q$, $\varphi(i^x) = a^y \in E_k^q$ for some $a \in B_i(A_k)$ and y. Then by Equation 5, under the allocation $M(\varphi, p^q)$, each $i \in N_k$ is assigned the objects in $B_i(A_k)$ with a probability of $d_{k-1}(i) + \lambda_k$. Therefore, $M(\varphi, p^q) \in f^{\text{EPS}}(p)$.

To finish the proof, it remains to show that for any $\sigma \in \mathscr{O}(p, q)$ and $\mu \in f^{\text{SD}}(\sigma, p^q)$, φ and μ are welfare equivalent.[49] Assume to the contrary, φ and μ are not welfare equivalent. Let i^x be the first agent in N^q who is not indifferent between φ and μ. That is, we do not have $\mu(i^x)\ I_i^q \varphi(i^x)$. Moreover, $\mu(j^y)\ I_j^q \varphi(j^y)$ for any $j^y \in N^q$ with $\sigma^{-1}(j^y) < \sigma^{-1}(i^x)$. It follows from the definition of serial dictatorships that $\mu(i^x)\ P_i^q \varphi(i^x)$. Suppose that $\mu(i^x) = a^y$ and $i^x \in N_k^q$. Then $aP_i b$ for any $b \in B_i(A_k)$. Note that, if $k(i) < \ell < k$, then $i \notin N_\ell$, and we have $B_i(A_\ell) \setminus E_\ell \neq \emptyset$, since otherwise

$$\frac{|B_{N_\ell \cup \{i\}}(A_\ell)| - \sum_{j \in N_\ell \cup \{i\}} d_{\ell-1}(j)}{|N_\ell \cup \{i\}|} = \frac{|E_\ell| - \sum_{j \in N_\ell \cup \{i\}} d_{\ell-1}(j)}{|N_\ell \cup \{i\}|} < \lambda_\ell.$$

Therefore, $B_i(A_{\ell+1}) \subseteq B_i(A_\ell)$. It follows that for any ℓ with $k(i) < \ell \leq k$, we have $aP_i b$ for any $b \in B_i(A_\ell)$, and hence $a \notin A_\ell$. This implies $a \in E_{k'}$ for some k' such that $1 \leq k' \leq k(i)$.

By the construction, all the objects $\cup_{\ell=1}^{k(i)} E_\ell^q$ are assigned to the agents $\cup_{\ell=1}^{k(i)} N_\ell^q$ under φ. In addition, $x > z$ for any $j^z \in \cup_{\ell=1}^{k(i)} N_\ell^q$, and $i^x \notin \cup_{\ell=1}^{k(i)} N_\ell^q$. Then, $\mu(i^x) = a^y \in E_{k'}^q \subseteq \cup_{\ell=1}^{k(i)} E_\ell^q$ implies the existence of some $j^z \in \cup_{\ell=1}^{k(i)} N_\ell^q$ such that $\mu(j^z) \notin \cup_{\ell=1}^{k(i)} E_\ell^q$. Let $j^z \in N_{k''}^q$, where $1 \leq k'' \leq k(i)$, and $\mu(j^z) = b^w$. Then $b \notin E_\ell$ for any $\ell \in \{1, \dots, k(i)\}$. This first implies $b \in A_{k''}$. However, since $x > z$ and $\sigma^{-1}(j^z) < \sigma^{-1}(i^x)$, we have $\mu(j^z)\ I_j^q \varphi(j^z)$, and hence $b \in B_j(A_{k''}) \subseteq E_{k''}$, which leads to a contradiction.

A.5 A Formal Definition of PST and Proof of Proposition 2

A.5.1 Definition of PST

Consider any $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HET}}$. In any step $k \geq 1$ of the allocation procedure, the set of remaining agents is N_k, the set of remaining objects is A_k, the remaining demand of each $i \in N_k$ is $r_k(i)$, and the remaining share of each $a \in A_k$ is $r_k(a)$, where $N_1 = N$,

[49] Specifically, it is easy to see that when φ and μ are welfare equivalent, $M(\varphi, p^q)$ and $M(\mu, p^q)$ are welfare equivalent. Then it can be shown that the ordinal fairness of $M(\varphi, p^q)$ implies $M(\mu, p^q)$ is also ordinally fair, and hence $M(\mu, p^q) \in f^{\text{EPS}}(p)$.

$A_1 = A$, $r_1(d) = 1$ for all $d \in N \cup A$.

Let every $i \in N_k$ point to the object in $b_i(A_k)$, and every $a \in A_k \cap A(p)$ point to $e_p^{-1}(a)$ if $e_p^{-1}(a) \in N_k$. There are two possible cases:

1. There exists a cycle. Pick any cycle, and denote C_N and C_A as the set of agents and the set of objects in the cycle, respectively. Let $\lambda_k = \min\{r_k(d) : d \in C_N \cup C_A\}$. Then each $i \in C_N$ is assigned λ_k of the object in $b_i(A_k)$. For each $i \in N_k$, let $r_{k+1}(i) = r_k(i) - \lambda_k$ if $i \in C_N$, and $r_{k+1}(i) = r_k(i)$ otherwise. For each $a \in A_k$, let $r_{k+1}(a) = r_k(a) - \lambda_k$ if $a \in C_A$, and $r_{k+1}(a) = r_k(a)$ otherwise.

2. There does not exist a cycle. Then every remaining agent $i \in N_k$ consumes her favorite remaining object at the rate $s_k(i)$. These rates are defined by the following two conditions: (1) if $i \in N_k \setminus N(p)$, $s_k(i) = 1$; (2) if $i \in N_k \cap N(p)$, $s_k(i) = 1 + \sum_{j \in N_k : b_j(A_k) = \{e_p(i)\}} s_k(j)$. For each $a \in A_k$, let

$$\lambda_k(a) = \frac{r_k(a)}{\sum_{i \in N_k : b_i(A_k) = \{a\}} s_k(i)}$$

if $\left\{i \in N_k : b_i(A_k) = \{a\}\right\} \neq \emptyset$, and $\lambda_k(a) = \infty$ otherwise. For each $i \in N_k$, let

$$\lambda_k(i) = \frac{r_k(i)}{s_k(i)}.$$

Define

$$\lambda_k = \min\{\lambda_k(d) : d \in N_k \cup A_k\}.$$

Then in step k, each $i \in N_k$ is assigned $s_k(i) \cdot \lambda_k$ of the object in $b_i(A_k)$. For each $i \in N_k$, let $r_{k+1}(i) = r_k(i) - s_k(i) \cdot \lambda_k$. For each $a \in A_k$, let $r_{k+1}(a) = r_k(a) - \lambda_k \cdot \sum_{i \in N_k : b_i(A_k) = \{a\}} s_k(i)$.

In either case, let $N_{k+1} = \{i \in N_k : r_{k+1}(i) > 0\}$ and $A_{k+1} = \{a \in A_k : r_{k+1}(a) > 0\}$. The procedure terminates in step $\bar{k}$ if $N_{\bar{k}+1} = \emptyset$.

A.5.2 Proof of Proposition 2

Let $p = (N, A, R, \succeq) \in \mathscr{P}_{\text{HET}}$ with $|N| = |A| = n$. If $N(p) = N$, then it is straightforward to show that for any q and any ordering of N^q, the deterministic allocation selected by TTC for $\tilde{p}^q$ generates the random allocation $f^{\text{PST}}(p)$ for p.

Assume $N(p) \neq N$. Consider the allocation procedure in the definition of PST, which terminates in some step $\bar{k}$. Define a function τ such that for every $k \in \{1, \ldots, \bar{k}\}$, $\tau(k) = 0$ if there exists a cycle in step k, and $\tau(k) = 1$ otherwise. For any $i \in N \setminus N(p)$, since she is not involved in any cycle and her consuming rate is always 1, we have $\sum_{k=1}^{\bar{k}} \tau(k)\lambda_k = 1$, $\tau(\bar{k}) = 1$, and $i \in N_{\bar{k}}$. Then in each step $k < \bar{k}$ of the procedure, either at least one object is exhausted, or at least one existing tenant's demand is fully satisfied. So $\bar{k} \leq n + |N(p)|$. Let $q = (n!)^{n+|N(p)|}$. We show by induction that, for each $k \in \{1, \ldots, \bar{k}\}$, $\lambda_k q$ is an integer, and $r_k(d)q$ is an integer for all $d \in N_k \cup A_k$. For $k = 1$, $r_1(d) = 1$ for all $d \in N_1 \cup A_1$. Moreover, $\lambda_1 = 1$ if $\tau(1) = 0$, and $\lambda_1 = \frac{1}{\sum_{i \in N_1 : b_i(A_1) = \{a\}} s_1(i)}$ for some $a \in A_1$ if $\tau(1) = 1$. As we will show below,[50] in any step, the sum of the rates of the agents who consume the same object cannot exceed n. Therefore, $\lambda_1(n!)$ is an integer. Assume that $\lambda_{k-1}(n!)^{k-1}$ is an integer, and $r_{k-1}(d)(n!)^{k-1}$ is an integer for all $d \in N_{k-1} \cup A_{k-1}$, where $1 < k \leq \bar{k}$. Then it is straightforward to show that $r_k(d)(n!)^{k-1}$ is an integer for all $d \in N_k \cup A_k$. Since $\lambda_k = \frac{r_k(d)}{m}$ for some $d \in N_k \cup A_k$ and integer $m \in \{1, \ldots, n\}$, $\lambda_k(n!)^k$ is an integer. Therefore, given that $\bar{k} \leq n + |N(p)|$ and $q = (n!)^{n+|N(p)|}$, for each $k \in \{1, \ldots, \bar{k}\}$, $\lambda_k q$ is an integer, and $r_k(d)q$ is an integer for all $d \in N_k \cup A_k$.

With respect to each step $k \in \{1, \ldots, \bar{k}\}$ of PST, we construct a subproblem of $\tilde{p}^q$, in which the number of parts of each $i \in N_k$ is $r_k(i)q$ and the number of parts of each $a \in A_k$ is $r_k(a)q$. Specifically, define $\tilde{p}_k^q = (N_k^q, A_k^q, \tilde{R}^q|_{N_k^q}, \tilde{\succeq}^q|_{A_k^q})$ such that

- $N_k^q = \left\{i^x : i \in N_k, \sum_{0 \leq \ell < k} \tau(\ell)\lambda_\ell q + 1 \leq x \leq \sum_{0 \leq \ell < k} \tau(\ell)\lambda_\ell q + r_k(i)q\right\}$, and
- $A_k^q = \left\{a^x : a \in A_k, q - r_k(a)q + 1 \leq x \leq q\right\}$,

where $\tau(0) = \lambda_0 = 0$. Then $\tilde{p}_1^q = \tilde{p}^q$, and $\{\tilde{p}_k^q\}_{k=1}^{\bar{k}}$ is a "decreasing" sequence of subproblems in the sense that $N_{k+1}^q \subseteq N_k^q$ and $A_{k+1}^q \subseteq A_k^q$ for each $k \in \{1, \ldots, \bar{k} - 1\}$. We further define $\tilde{p}_{\bar{k}+1}^q$ as the "empty" subproblem in which $N_{\bar{k}+1}^q = A_{\bar{k}+1}^q = \emptyset$.

Let σ be an ordering of the agents N^q. To prove the proposition, it is sufficient to show the following for each $k \in \{1, \ldots, \bar{k}\}$:

(1) Given $\tilde{p}_k^q$, we can reach $\tilde{p}_{k+1}^q$ by iteratively removing a particular sequence of trading cycles that we construct.

(2) If $i \in N_k$ is assigned m of a in step k of PST, then mq parts of i are assigned parts of a in those trading cycles.

[50]See Claim 5.

Since the outcome of TTC is independent of the order that trading cycles are removed, (1) implies that $\mu = f^{\text{TTC}}(\sigma, \tilde{p}^q)$ is completely characterized by the trading cycles that we construct. Given this, (2) implies $M(\mu, \tilde{p}^q) = f^{\text{PST}}(p)$.

Let $k \in \{1, \ldots, \bar{k}\}$. We first consider the case of $\tau(k) = 1$. Construct one cycle corresponding to each $i \in N_k$ as follows: let every $j \in N_k$ point to the object in $b_j(A_k)$, every $a \in A_k \cap A(p)$ with $e_p^{-1}(a) \in N_k$ point to $e_p^{-1}(a)$, and all the other objects in A_k point to i. Since $\tau(k) = 1$ and there is no cycle among the existing tenants and their endowments, there must exist a unique cycle with distinct objects and distinct agents which include i. We denote it as $C(i) = (i_1, a_1, \ldots, i_s, a_s)$, where $s \geq 1$, $i_1 = i$, each agent (object) in the list points to the object (agent) next to it, and a_s points to i_1. Moreover, we have

- $a_s \neq e_p(i)$ when $i \in N(p)$, and
- for every $t \neq s$, $a_t \in A(p)$ and $a_t = e_p(i_{t+1})$.

Let $C_N(i)$ and $C_A(i)$ be the set of agents and the set of objects in the cycle $C(i)$, respectively.

By the construction of the set of cycles $\{C(i) : i \in N_k\}$ and their properties discussed above,

- for each $i \in N_k \setminus N(p)$, $i \notin C_N(j)$ for any $j \in N_k \setminus \{i\}$,
- for each $i \in N_k \cap N(p)$, if $j \in N_k \setminus \{i\}$, $B_j(A_k) = \{e_p(i)\}$,[51] and $j \in C_N(j')$, then $j' \neq i$, $i \in C_N(j')$ and there does not exist $j'' \in N_k \setminus \{i, j\}$ such that $B_{j''}(A_k) = \{e_p(i)\}$ and $j'' \in C_N(j')$, and
- for each $i \in N_k \cap N(p)$, if $i \in C_N(j)$ for some $j \neq i$, then there exists $j' \in C_N(j)$ such that $B_{j'}(A_k) = \{e_p(i)\}$.

This implies that

- for each $i \in N_k \setminus N(p)$, $|\{j \in N_k : i \in C_N(j)\}| = 1$, and
- for each $i \in N_k \cap N(p)$, $|\{j \in N_k : i \in C_N(j)\}| = 1 + \sum_{j \in N_k : b_j(A_k) = \{e_p(i)\}} |\{j' \in N_k : j \in C_N(j')\}|$.

[51]Note that an existing tenant's endowment cannot be exhausted before her demand is fully satisfied, due to the specification of her consuming rate.

Thus, by comparing with the specification of the consuming rates $\{s_k(i)\}_{i \in N_k}$ in the definition of PST, we know that each agent's rate is equal to the number of the cycles in which she appears:

Claim 4. *For each $i \in N_k$, $s_k(i) = |\{j \in N_k : i \in C_N(j)\}|$.*

It follows that the sum of the consuming rates of those agents who consume the same object is equal to the number of cycles in which the object appears:

Claim 5. *For each $a \in A_k$, $\sum_{i \in N_k : b_i(A_k) = \{a\}} s_k(i) = |\{i \in N_k : a \in C_A(i)\}|$.*

Next, we will construct a sequence of trading cycles based on $\{C(i)\}_{i \in N_k}$ such that $\tilde{p}^q_{k+1}$ can be reached from $\tilde{p}^q_k$ by iteratively removing them.

Given $i \in N_k$ and $C(i) = (i_1, a_1, \ldots, i_s, a_s)$, where $i_1 = i$, we say $(i_1^{x_1}, a_1^{y_1}, \ldots, i_s^{x_s}, a_s^{y_s})$ is a *cycle of type $C(i)$* if

- $a_t^{y_t} \in A^q_k$ for every $t \in \{1, \ldots, s\}$,
- $\sum_{0 \le \ell < k} \tau(\ell)\lambda_\ell q + 1 \le x_1 \le \sum_{0 \le \ell \le k} \tau(\ell)\lambda_\ell q$, and
- $\sum_{0 \le \ell \le k} \tau(\ell)\lambda_\ell q + 1 \le x_t \le \sum_{0 \le \ell < k} \tau(\ell)\lambda_\ell q + r_k(i_t)q$ for every $t \neq 1$.

For each $x \in \left\{\sum_{0 \le \ell < k} \tau(\ell)\lambda_\ell q + 1, \ldots, \sum_{0 \le \ell \le k} \tau(\ell)\lambda_\ell q\right\}$, we construct one cycle of every type $C(i)$ that includes i^x, through an iterative procedure. Consider first $x = \sum_{0 \le \ell < k} \tau(\ell)\lambda_\ell q + 1$. Note that in the subproblem $\tilde{p}^q_k$, at every object $a^y \in A^q_k$ such that there does not exist $i \in N_k$ with $e_p(i) = a$, the agents in $\{i^x : i \in N_k\}$ are ranked equally, and they are ranked higher than any other agent. Let the set of remaining agents and the set of remaining objects at round 1 be N^q_k and A^q_k, respectively. In general, at round m, where $1 \le m \le |N_k|$, if i^x is the m-th agent among $\{j^x : j \in N_k\}$ according to σ, then we can find a cycle of type $C(i)$ such that

- i^x is in the cycle,
- if $j^y \neq i^x$ is in the cycle, then j^y is a remaining agent at round m, and there does not exist a remaining agent j^z at round m such that $z > y$, and
- if a^y is in the cycle, then a^y is a remaining object at round m, and there does not exist a remaining object a^z at round m such that $z < y$.

Remove the agents and the objects in the cycle constructed at round m, and proceed to the next round if $m < |N_k|$. By the construction, the cycle found at each round is a trading cycle in the reduced subproblem that is obtained by removing from $\tilde{p}_k^q$ all the cycles at the previous rounds. Then, we can continue and repeat this process, for $x = \sum_{0\le\ell<k}\tau(\ell)\lambda_\ell q + 2, \ldots, \sum_{0\le\ell\le k}\tau(\ell)\lambda_\ell q$. This leads to a sequence of trading cycles, including $\lambda_k q$ cycles of each type. Let $\hat{N}_k^q$ and $\hat{A}_k^q$ denote all the agents and all the objects in these trading cycles, respectively.

For each $a \in A_k$, by Claim 5,

$$\left|\{a^x : a^x \in \hat{A}_k^q\}\right| = \lambda_k q \left|\{i \in N_k : a \in C_A(i)\}\right| = \lambda_k q \sum_{i\in N_k : b_i(A_k)=\{a\}} s_k(i).$$

This implies that, after removing the trading cycles from $\tilde{p}_k^q$, the set of remaining objects is A_{k+1}^q. On the other hand, for each $i \in N_k$, $\left\{i^x : \sum_{0\le\ell<k}\tau(\ell)\lambda_\ell q + 1 \le x \le \sum_{0\le\ell\le k}\tau(\ell)\lambda_\ell q\right\} \subseteq \hat{N}_k^q$. If $s_k(i) > 1$, then by Claim 4, there are additional $\lambda_k q(s_k(i)-1)$ parts of i that are included in the trading cycles:

$$\left\{i^x : \sum_{0\le\ell<k}\tau(\ell)\lambda_\ell q + r_k(i)q - \lambda_k q(s_k(i)-1) + 1 \le x \le \sum_{0\le\ell<k}\tau(\ell)\lambda_\ell q + r_k(i)q\right\} \subseteq \hat{N}_k^q.$$

This implies that, after removing the trading cycles from $\tilde{p}_k^q$, the set of remaining agents is N_{k+1}^q. Moreover, for each $i \in N_k$, $|\{i^x : i^x \in \hat{N}_k^q\}| = s_k(i)\lambda_k q$. That is, if $b_i(A_k) = \{a\}$, then $s_k(i)\lambda_k q$ parts of i are assigned parts of a. This finishes the proof for the case of $\tau(k) = 1$.

Finally, we consider the case that there exists a cycle among the existing tenants and their endowments in step k of PST, i.e., $\tau(k) = 0$. Let the cycle selected by PST be $C = (i_1, a_1, \ldots, i_s, a_s)$, where $s \ge 1$, each agent (object) in the list points to the object (agent) next to it, and a_s points to i_1. For each $t \in \{1, \ldots, s\}$, let $x_t = \max\{x : i_t^x \in N_k^q\}$ and $y_t = \min\{x : a_t^x \in A_k^q\}$. Let $c_z = \left(i_1^{x_1-z}, a_1^{y_1+z}, \ldots, i_s^{x_s-z}, a_s^{y_s+z}\right)$. Then $(c_z)_{z=0}^{\lambda_k q-1}$ is a sequence of trading cycles: c_0 is a trading cycle for $\tilde{p}_k^q$; after removing the trading cycles $c_0, \ldots, c_z$ from $\tilde{p}_k^q$, where $0 \le z < \lambda_k q - 1$, c_{z+1} is a trading cycle in the reduced subproblem. It is clear that for each agent i in the cycle C with $b_i(A_k) = \{a\}$, $\lambda_k q$ parts of i are assigned parts of a in the trading cycles $(c_z)_{z=0}^{\lambda_k q-1}$. Moreover, after all of these trading cycles are removed, the reduced subproblem is $\tilde{p}_{k+1}^q$.

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